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Local Polynomial Convexity of the Unfolded Whitney Umbrella in $\mathbb{C}^2$

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The paper considers a class of Lagrangian surfaces in $\mathbb{C}^2$ with isolated singularities of the unfolded Whitney umbrella type. We prove that generically such a surface is locally polynomially convex near a singular point of this kind.

1 Introduction

Owing to its deep relation to the approximation problems, pluripotential theory, and Banach algebras (see, for instance, [2, 31] for a detailed discussion), polynomial convexity of real submanifolds of $\mathbb{C}^n$ is a well-studied subject in complex analysis. Gromov [18] found remarkable connections between the polynomial (or the holomorphic disc) convexity of real manifolds and global rigidity of symplectic structures. In the present work, we prove that a generic Lagrangian surface in $\mathbb{C}^2$ is polynomially convex near an isolated singularity which is topologically an unfolded Whitney umbrella. This study is inspired by the work of Givental [17], where he proved, in particular, that a compact real surface S admits a smooth map $\iota : S \to \mathbb{C}^2$, isotropic with respect to the standard symplectic structure on $\mathbb{C}^2$, such that the singularities of ι are isolated and either self-intersections

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or unfolded Whitney umbrellas. More precisely, if we denote the standard coordinates in $\mathbb{C}^2$ by $z = x + iy$ and $w = u + iv$, then

$$\omega = dx \wedge dy + du \wedge dv$$

is the standard symplectic form on $\mathbb{C}^2$. A smooth map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is called *symplectic* if $\phi^*\omega = \omega$. Since such a map is a local diffeomorphism, we call it a (local) *symplectomorphism*. A smooth map $\iota : S \to (\mathbb{C}^2, \omega)$ from a smooth real surface S is called *isotropic* if $\iota^*\omega = 0$. Givental [17] showed that near a generic point $p \in S$, which is an isolated singular point of ι of rank 1, the map

$$\pi : \mathbb{R}^2_{(t,s)} \to \mathbb{R}^4_{(x,u,y,v)} : (t, s) \to \left(ts, \frac{2t^3}{3}, t^2, s\right) \quad (1)$$

is a local normal form for ι. In particular, this means that there exists a local symplectomorphism near $\iota(p)$ sending $\iota(S)$ onto a neighborhood of the origin in $\Sigma := \pi(\mathbb{R}^2)$. The set Σ, as well as $\iota(S)$ near $\iota(p)$, is called the *unfolded (or open) Whitney umbrella*. Our main result is the following.

Theorem 1. Suppose $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is either a generic real-analytic symplectomorphism near the origin or the identity map. Then there exists a neighborhood of the point $\phi(0)$ in the surface $\phi(\Sigma)$ with compact polynomially convex closure. □

The case where ϕ is the identity map is considered separately since it is not generic. This implies that the Whitney umbrella Σ is polynomially convex near the origin. This theorem also holds under weaker assumptions, namely, if ϕ is a generic local real-analytic diffeomorphism and $D\phi(0)$, the differential of ϕ at zero, is symplectic, or if ϕ is a C^∞-smooth symplectomorphism with the jet at the origin satisfying some additional assumptions; see Section 5 for details.

Denote by $\mathbb{B}(p, r)$ the open Euclidean ball of $\mathbb{C}^2$ of radius $r > 0$ centered at p. As an application of Theorem 1, we obtain the following result.

Corollary 1. Let ϕ be as in Theorem 1. Then for $\varepsilon > 0$ sufficiently small, any continuous function on $\phi(\Sigma) \cap \mathbb{B}(\phi(0), \varepsilon)$ can be uniformly approximated by holomorphic polynomials. □

It will be shown in Section 4 that the genericity assumption of Theorem 1 imposes restrictions only on the 2-jet of ϕ at the origin. More precisely, it suffices to require that such a jet does not lie in a real-algebraic submanifold of codimension 2 (after the standard identification of the space of 2-jets at the origin with the Euclidean space). Our approach is based on the observation that $\phi(\Sigma)$ is contained in the zero locus set M of a strictly plurisubharmonic function with a unique critical point at the origin. Hence M is a strictly pseudoconvex hypersurface smooth everywhere except the origin. This allows us to consider the characteristic foliation induced on $\phi(\Sigma)$ by the embedding $\phi(\Sigma) \hookrightarrow M$. The origin is a unique singular point for this foliation. It follows by the Hopf lemma that if f is a holomorphic disc with boundary attached to $\phi(\Sigma)$, then its boundary is transverse to the leaves of the characteristic foliation at every point different from the origin. Suppose now that the structure of leaves of the characteristic foliation near the origin is topologically the same as the phase portrait of a dynamical system near a saddle stationary point on the plane. Then the boundary of f will *touch* a leaf of the characteristic foliation, proving that such a holomorphic disc does not exist. This observation suggests a strategy for the proof of our main result. The proof consists of two parts.

First, we use Oka's Characterization Theorem for hulls [25], developed and adapted to the case under consideration in the work of Stolzenberg [29], Duval [12], and Jöricke [22]. This enables us to generalize the aforementioned argument and prove polynomial convexity of $\phi(\Sigma)$ near the origin under the assumption that the phase portrait of the characteristic foliation is topologically a saddle (Sections 2 and 3). The remainder of the paper (Sections 4–7) is devoted to the study of the characteristic foliation near the origin. In Section 4, we write explicitly a 5-jet of the corresponding dynamical system on the plane; the origin is a stationary point with a high order of degeneracy. At the end of this section, we describe explicitly the genericity assumption on the 2-jet of ϕ. Section 5 is expository: for the reader's convenience we recall relevant tools from the local theory of dynamical systems; in particular, we explain where the real analyticity assumption comes from. In Sections 6 and 7, we give a complete topological description of the phase portrait of the aforementioned dynamical system proving that it is a saddle.

The problem remains open to determine local polynomial convexity for nongeneric Whitney umbrellas as we have no counterexamples to Theorem 1 if the genericity assumption is dropped. Our method relies on the properties of the phase portrait of the dynamical system associated with the characteristic foliation near the umbrella, and cannot be applied if some specific terms in the low-order jets at the origin of the map ϕ vanish. On the other hand, in applications to topological properties of surfaces the generic situation is often sufficient. Furthermore, our method works in some non-generic cases, for instance, for the standard umbrella Σ (this case is treated separately in Sections 4 and 6).

Convexity (polynomial, rational or holomorphic) of a Lagrangian or totally real manifold E embedded into $\mathbb{C}^n$ have been studied by several authors (see, for instance, [1, 2, 11, 13, 16, 18, 21, 31]). It is well known that the local polynomial convexity can fail near points where E is not totally real. In the complex dimension $n=2$, the tangent space of E is a complex line, so such points are called *complex*; generically, these points are isolated in E. The complex geometry of these points is well understood by now. There are three types of generic complex points: elliptic, hyperbolic, and parabolic (see, for instance, [2, 31]), and the local polynomial convexity depends on the type. Bishop [5] and Kenig and Webster [24] proved that a neighborhood of an elliptic point in E has a nontrivial hull. On the other hand, Forstnerič and Stout [15] proved that E is locally polynomially convex near a hyperbolic point. The parabolic case is intermediate and in general both possibilities occur. This case was studied by Jöricke [22, 23]. These results and their development have several important applications, in particular, to the problem of complex and symplectic filling and topological classification of 3-contact structures.

In general, a compact real surface does not admits a Lagrangian or totally real embedding into $\mathbb{C}^2$; for instance, torus is the only compact orientable real surface admitting a Lagrangian embedding into $\mathbb{C}^2$. By comparison, Givental's result is quite general as it applies to *all* compact surfaces. This makes it natural to study self-intersections and Whitney umbrellas on immersed Lagrangian manifolds in analogy with local analysis of real surfaces near complex points. Currently, only few results are obtained in this direction.

The present work is the first step in the study of the most general case where Whitney umbrellas arise. Our result implies that local convexity properties near a generic real analytic Lagrangian deformation of the standard Whitney umbrella are similar to those of a hyperbolic point. This is a necessary step leading toward understanding the global geometry of immersed Lagrangian manifolds containing Whitney umbrellas.

2 Geometry of Whitney Umbrellas

The map $\pi : \mathbb{R}^2_{(t,s)} \to \mathbb{R}^4_{(x,u,y,v)}$ given by (1) is a smooth homeomorphism onto its image, nondegenerate except at the origin, where the rank of π equals one. It satisfies $\pi^*\omega \equiv 0$, and so Σ is a Lagrangian submanifold of $(\mathbb{C}^2, \omega)$ with an isolated singular point at the origin. Thus,

$$\Sigma = \left\{(z, w) \in \mathbb{C}^2 : x = ts,\ u = \frac{2t^3}{3},\ y = t^2,\ v = s;\ t, s \in \mathbb{R}\right\}.$$

The crucial role in our approach is played by an auxiliary real hypersurface M defined by

$$M = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) = x^2 - yv^2 + \tfrac{9}{4}u^2 - y^3 = 0\}. \tag{2}$$

Clearly, Σ is contained in M. Note that the hypersurface M is smooth away from the origin, and strictly pseudoconvex in $\mathbb{B}(0, \varepsilon) \setminus \{0\}$ for ε sufficiently small.

Suppose now that $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is a local smooth diffeomorphism near the origin such that its linear part $D\phi(0)$ at the origin is a symplectic map. Without loss of generality, we may assume that $\phi(0) = 0$. The standard symplectic structure on $\mathbb{C}^2$ is given by the matrix

$$\Omega = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where I_2 denotes the identity matrix on $\mathbb{R}^2$. Similarly, we write

$$D\phi(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3}$$

The condition that $D\phi(0)$ is symplectic means that $(D\phi(0))^t\, \Omega\, D\phi(0) = \Omega$ (where t stands for matrix transposition). Therefore, the real (2×2)-matrices A, $B = (b_{jk})$, C and $D = (d_{jk})$ satisfy

$$A^t D - C^t B = I_2, \quad A^t C = C^t A, \quad D^t B = B^t D. \tag{4}$$

The standard complex structure of $\mathbb{C}^2$ in real coordinates is given by the matrix

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

which corresponds to multiplication by i. We perform an additional complex linear change of coordinates ψ. Let $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by the 4×4

matrix

$$\begin{pmatrix} D^t & -B^t \\ B^t & D^t \end{pmatrix}. \tag{5}$$

This matrix commutes with J and so gives rise to a nondegenerate complex linear map in $\mathbb{C}^2$. Let

$$\Sigma' = \psi \circ \phi(\Sigma),$$

and

$$M' = (\psi \circ \phi)(M).$$

The differential at the origin of the composition $\psi \circ \phi$ is given by

$$D(\psi \circ \phi)(0) = \begin{pmatrix} I_2 & 0 \\ E & G \end{pmatrix}, \tag{6}$$

where we used identities (4) to simplify the matrix. Further, a direct calculation shows that

$$G = (g_{kj}) = B^t B + D^t D, \tag{7}$$

and therefore, the matrix G is symmetric with positive entries in the main diagonal. The determinant

$$\Delta = g_{11} g_{22} - g_{12}^2 \tag{8}$$

of G coincides with that of the matrix in (5) corresponding to a $\mathbb{C}$-linear map of $\mathbb{C}^2$. Hence Δ is also positive. Let $\rho' = \rho \circ (\psi \circ \phi)^{-1}$, and

$$\Omega' = \{(z', w') \in \mathbb{C}^2 : \rho'(z', w') < 0\}. \tag{9}$$

It follows from (2) and (6) that

$$\rho'(z', w') = x'^2 + \tfrac{9}{4} u'^2 + o(|(z', w')|^2). \tag{10}$$

In particular, the function ρ' is strictly plurisubharmonic in a neighborhood of the origin, and the hypersurface M' is strictly pseudoconvex in a punctured neighborhood of the origin.

Lemma 1. The polynomial hull of the set $\mathbb{B}(0,\varepsilon)\cap\Sigma'$ for sufficiently small $\varepsilon>0$ is contained in $\overline{\Omega'\cap\mathbb{B}(0,\varepsilon)}$. □

Proof. Choose $\varepsilon>0$ small enough such that ρ' is strictly plurisubharmonic in $\mathbb{B}(0,\varepsilon)$. The polynomially convex hull of $\mathbb{B}(0,\varepsilon)\cap\Sigma'$ is contained in $\overline{\mathbb{B}(0,\varepsilon)}$. By a classical result (see, for instance, [20]), the polynomially convex hull of $\overline{\mathbb{B}(0,\varepsilon)\cap\Sigma'}$ coincides with its hull with respect to the family of functions plurisubharmonic in $\mathbb{B}(0,\varepsilon)$. Since for any point p in $\overline{\mathbb{B}(0,\varepsilon)}\setminus\bar{\Omega}'$, we have $\rho'(p)>0$, the assertion of the lemma follows. ■

3 Characteristic Foliation and Polynomial Convexity

In this section, we explain the strategy of the proof of Theorem 1.

3.1 Characteristic foliation

Let X be a totally real surface embedded into a real hypersurface Y in $\mathbb{C}^2$. Define on X a field of lines determined at every $p\in X$ by

$$L_p = T_pX\cap H_pY,$$

where $H_pY = T_pY\cap J(T_pY)$ denotes the complex tangent line to Y at the point p and J denotes the standard complex structure of $\mathbb{C}^2$. Integral curves, that is, curves that are tangent to L_p at each point p, of this line field define a foliation on X. It is called the *characteristic foliation* of X.

We consider the characteristic foliation of $\Sigma\setminus\{0\}\subset M$ and $(\psi\circ\phi)(\Sigma)\setminus\{0\}\subset(\psi\circ\phi)(M)$. Characteristic foliations are invariant under biholomorphisms. Therefore, in order to study the characteristic foliation on $\phi(\Sigma)$ with respect to $\phi(M)$, it is sufficient to study the characteristic foliation of $\Sigma'=\psi\circ\phi(\Sigma)$ with respect to M'.

Recall that a *rectifiable arc* is a homeomorphic image of an interval under a Lipschitz map. Our ultimate goal is to prove the following.

Proposition 1. There exist $\varepsilon>0$ small enough and two rectifiable arcs γ_1 and γ_2 in $\Sigma'\cap\mathbb{B}(0,\varepsilon)$ passing through the origin with the following properties:

(i) γ_j are smooth at all points except, possibly, the origin;

(ii) $\gamma_1\cap\gamma_2=\{0\}$;

(iii) if K is a compact subset of $\Sigma' \cap \mathbb{B}(0, \varepsilon)$ and is not contained in $\gamma_1 \cup \gamma_2$, then there exists a leaf γ of the characteristic foliation on Σ' such that $K \cap \gamma \neq \varnothing$ but K does not meet both sides of γ. □

We point out that by (i) and (ii) the union $\gamma_1 \cup \gamma_2$ does not bound any subdomain with the closure compactly contained in $\Sigma' \cap \mathbb{B}(0, \varepsilon)$.

The proof of the proposition will be given in Sections 4–7. Considering pull-back of the characteristic foliation by $\psi \circ \phi \circ \pi$ we obtain a smooth vector field in a neighborhood of the origin in $\mathbb{R}^2_{(t,s)}$ with the stationary point at the origin. The study of its integral curves is based on the local theory of dynamical systems and can be read independently from the rest of the paper.

Assuming Proposition 1, we now prove our main results. The proof is based on the argument due to Duval [12] and Jöricke [22, 23]. Suppose that ϕ satisfies the assumptions of Theorem 1, and $\Sigma' = (\psi \circ \phi)(\Sigma)$. First we establish nonexistence of holomorphic discs attached to Σ' near the Whitney umbrella. In what follows we denote by Δ the unit disc of $\mathbb{C}$. By a holomorphic disc we mean a map $f : \Delta \to \mathbb{C}^2$ holomorphic in Δ and continuous on $\bar{\Delta}$. As usual, by its boundary we mean the restriction $f|_{\partial\Delta}$; we identify it with its image $f(\partial\Delta)$.

Corollary 2. There exists $\delta > 0$ with the following property: a holomorphic disc $f : \Delta \to \mathbb{B}(0, \delta)$ with the boundary attached to Σ', that is, satisfying $f(\partial\Delta) \subset \Sigma'$, is constant. □

Before we proceed with the proof, we recall some basic notions. Let $U \subset \mathbb{R}^n$ be a domain and N be a real submanifold of dimension d in U. As usual, denote by $\mathcal{D}(U)$ the space of test-functions on U. The *current of integration* $[N]$ corresponding to N is a continuous linear form on the space $\mathcal{D}^d(U)$ of differential forms of degree d with coefficients in $\mathcal{D}(U)$ defined by

$$[N](\psi) = \int_N \psi, \quad \forall \psi \in \mathcal{D}^d(U). \tag{11}$$

The current $[N]$ may be well-defined even when N has some singularities provided that the behavior of N near its singular locus is not too bad. For instance, the current of integration over a complex analytic set or a rectifiable curve is well-defined, see [7, 14, 19, 31]. The exterior derivative $d[N]$ is then defined by duality: $d[N](\psi) := (-1)^{n-d+1}[N](d\psi)$.

Proof. Let $\varepsilon > 0$ be given by Proposition 1. Without loss of generality, we may assume that ε is such that the function ρ' in (10) is strictly plurisubharmonic in the ball $\mathbb{B}(0, 2\varepsilon)$. Set $\delta = \varepsilon/2$. Suppose that there exists a nonconstant holomorphic disc $f : \Delta \to \mathbb{B}(0, \delta)$ with boundary glued to Σ'. The function $\rho' \circ f$ is subharmonic in the unit disc, so the maximum principle implies that $f(\Delta)$ is contained in $\Omega' = \{\rho' < 0\}$. The proof consists of two parts.

(1) First we show that the boundary of f is not contained in $\gamma_1 \cup \gamma_2$. Arguing by contradiction, assume that $f(\partial\Delta) \subset \gamma_1 \cup \gamma_2$. The image $V := f(\Delta)$ is a complex one-dimensional analytic subset of Ω' and its boundary $bV := \bar{V} \backslash V$ is contained in $\gamma_1 \cup \gamma_2$. Since the arcs γ_j are rectifiable, it follows by the well-known results [7, 19, 31] that two cases can occur. The first possibility is that the closure $\bar{V}$ is a complex one-dimensional analytic subset of $\mathbb{C}^2$ contained in $\mathbb{B}(0, \varepsilon)$. This is impossible since a closed complex analytic subset of positive dimension cannot be compactly contained in $\mathbb{C}^2$ (e.g., [7]). The second case is when the area of V is bounded, V defines the current of integration $[V]$ on $\mathbb{C}^2$, and $d[V] = -[bV]$ in the sense of currents. Since $d^2 = 0$ for currents, the current $[bV]$ is closed, that is, $d[bV](\psi) = 0$ for all $\psi \in \mathcal{D}(\mathbb{C}^2)$. Furthermore, there exists a closed subset E in bV of the Hausdorff 1-measure 0 such that the couple (V, bV) is a complex manifold with boundary in a neighborhood of every point in $bV \setminus E$. Then bV is the union of closed subarcs of the arcs γ_j. In particular, bV is not a closed curve and has nonempty boundary in $\mathbb{C}^2$. Let p be a boundary point of bV and U be a sufficiently small neighborhood of p such that $U \cap bV$ is an arc in U with the end p. Considering test-forms $\psi \in \mathcal{D}^1(U)$, we conclude by Stokes' formula that $d[bV] \neq 0$ in $\mathbb{C}^2$ since the Dirac mass at p appears in the exterior derivative: a contradiction.

(2) By the uniqueness theorem the set of points $f^{-1}(0)$ has measure zero on the unit circle. Since Σ' is totally real outside the origin, it follows by the boundary regularity theorem [7] that f is smooth (even real analytic) up to the boundary outside the pull-back $f^{-1}(0)$. Applying the Hopf lemma (see, for instance, [27]) to the subharmonic function $\rho' \circ f$ on Δ, we conclude that f is transverse to the hypersurface M' at every point different from the origin. Therefore, the complex line tangent to $f(\Delta)$ at a boundary point is transverse to the tangent complex line of M' at this point. In particular, the boundary $K := f(\partial\Delta)$ is transverse to the leaves of the characteristic foliation of Σ'. This contradicts Proposition 1. ■

3.2 Sweeping out the envelope by analytic curves

Given a compact set K, we denote by $\hat{K}$ its polynomially convex hull. We also recall two useful related notions. The *essential hull* $K^{\rm ess}$ of K is defined by

$$K^{\rm ess} = \overline{\hat{K} \setminus K},$$

and the *trace* $K^{\rm tr}$ of $K^{\rm ess}$ is the intersection

$$K^{\rm tr} = K^{\rm ess} \cap K.$$

A local maximum principle of Rossi [28, 31] states that if K is a compact set in $\mathbb{C}^n$, $E \subset \hat{K}$ is compact, U is an open subset of $\mathbb{C}^n$ that contains E, and if $f \in \mathcal{O}(U)$, then $||f||_E = ||f||_{(E\cap K)\cup\partial E}$, where the boundary of E is taken with respect to $\hat{K}$. By choosing $E = K^{\rm ess}$ and $U = \mathbb{C}^2$, we see that $K^{\rm ess}$ is contained in $\hat{K}^{\rm tr}$. Therefore, to prove that K is polynomially convex, it is enough to show that $K^{\rm tr}$ is empty.

Let

$$X = \Sigma' \cap \overline{\mathbb{B}(0,\varepsilon)}.$$

Then X is a closed disc, and the punctured disc $X \setminus \{0\}$ is real analytically and total really embedded into $\partial\Omega' \setminus \{0\}$, where Ω' is given by (9), and ε is such that Lemma 1 holds.

Proposition 2. The essential hull $X^{\rm ess}$ cannot intersect a leaf of a characteristic foliation at a totally real point of X without crossing it. □

This result is due to Duval [12] (see also Jöricke [22]) in the case where a totally real disc is contained in the boundary of a smoothly bounded strictly pseudoconvex domain of $\mathbb{C}^2$. A detailed exposition of the proof is contained in [31]. The proof, which is an application of Oka's method (developed also by Stolzenberg [29]), is purely local and works without any essential modification in our case where $\partial\Omega'$ admits an isolated singularity at the origin. For reader's convenience we sketch the main steps of this construction.

Step 1. *Oka's characterization theorem.* We will state all results for dimension 2 because we deal with this case only; for more general versions, see [29, 31].

Let $U \subset O$ be two open subsets of $\mathbb{C}^2$. Let $F : [0,1] \times U \to \mathbb{C}$ be a continuous function that for every $t \in [0,1]$ defines a nonconstant holomorphic function $f_t := F(t, \bullet)$ on U.

The zero locus of f_t,

$$V_t := \{p \in U : f_t(p) = 0\}, \quad t \in [0, 1],$$

is a purely one-dimensional complex analytic subset of U. Suppose that every V_t is also closed in O. Then we call V_t *an analytic curve in* O and call $\{V_t\}_{t\in[0,1]}$ a *continuous family of analytic curves* in O. The classical version of Oka's method is the following (see [31]):

Oka's Characterization Theorem. Let K be a compact subset of $\mathbb{C}^2$ and O be a neighborhood of $\hat{K}$. If $\{V_t\}$ is a continuous family of analytic curves in O such that V_0 intersects $\hat{K}$, but V_1 does not, then some V_t must intersect K. □

Many various versions of this fundamental principle are known. For us the following criterion is useful (cf. [11]): Let $\{V_t\}_{t\in[0,1]}$ be a continuous family of analytic curves in a neighborhood O of $\overline{\Omega' \cap \mathbb{B}(0,\varepsilon)}$ such that for all t the curves V_t do not intersect X^{tr} and V_1 does not intersect $\bar{\Omega}'$. Then the curves V_t do not intersect X^{ess}.

Indeed, since the essential hull X^{ess} is contained in $\hat{X}^{\mathrm{tr}}$ by Rossi's local maximum principle and $\hat{X}^{\mathrm{tr}}$ is contained in $\overline{\Omega' \cap \mathbb{B}(0,\varepsilon)}$ by Lemma 1, it suffices to apply Oka's theorem.

The first step of the construction is the following key technical tool of [12]:

Lemma 2. Let $p \in X \setminus \{0\}$ be an arbitrary point. Then p does not lie in X^{tr} if there exist two continuous families $\{V_t\}_{t\in[0,1]}$ and $\{W_t\}_{t\in[0,1]}$ of analytic curves in an open neighborhood O of $\overline{\Omega' \cap \mathbb{B}(0,\varepsilon)}$ with the following properties:

(i) V_0 and W_0 meet X transversely at p and with opposite signs of intersection;
(ii) for $t > 0$, the varieties V_t and W_t are disjoint from X^{tr};
(iii) V_1 and W_1 do not intersect $\bar{\Omega}'$. □

Duval's original result is stated for the $\mathcal{O}(\bar{G})$-hull of a smooth totally real surface $X \subset \partial G$, where $G \subset \mathbb{C}^2$ is a smoothly bounded strictly pseudoconvex domain. The proof is also valid in our situation. Indeed, in order to show that p does not belong to X^{ess}, it suffices to find a neighborhood U of p such that $\hat{X}$ does not intersect $U \setminus X$. Let $F, G : [0,1] \times O \to \mathbb{C}$ be the functions defining the families $\{V_t\}$, $\{W_t\}$ that satisfy conditions of the lemma. We use the notation $f_t = F(t, \bullet)$ and $g_t = G(t, \bullet)$. It follows from (i) that near p the functions f_0 and g_0 provide local holomorphic coordinates and the real surface X is defined near p by the equation $g_0 = h \circ f_0$. Here h is a C^2-diffeomorphism in a neighborhood of the origin in $\mathbb{C}$, fixing the origin and reversing the orientation, so that $|h_{\bar{\zeta}}(0)| > |h_\zeta(0)|$. Denote by $\tau\Delta_-$ the left semidisc of radius $\tau > 0$, that is, $\tau\Delta_- = \{\zeta \in \mathbb{C} : |\zeta| < \tau,\ \Re\zeta < 0\}$. For $\alpha \in \tau\Delta_-$ and a complex parameter a, consider the analytic curves C_a in O defined by the equation

$$(f_0 - a)(g_0 - h(a)) = \alpha h_{\bar{\zeta}}(a).$$

There exists $\tau > 0$ such that when the parameter a runs over a small neighborhood of the origin in $\mathbb{C}$ and α runs over $\tau\Delta_-$, the family $\{C_a\}$ fills out an open set $U \setminus X$ for a suitable neighborhood U of p. The proof due to [10, Lemma 1, pp. 584–585], is obtained by the linear approximation of h near a. One verifies two properties of the family C_a. First, given $\alpha \in \tau\Delta_-$ and a, the curve C_a avoids X. Second, for every point $q \in U \setminus X$, one can find suitable a and α such that C_a contains q.

Finally, we note that every curve C_a can be swept out of Ω' through a continuous family of analytic curves in O in accordance with Oka's characterization of hulls. Such a sweeping family of analytic curves is explicitly constructed in [12, pp. 110–111], using the defining functions f_t, g_t and the assumptions (ii) and (iii) of Lemma 2.

This shows that no point near p can be in X^{ess}, and therefore p does not belong to X^{tr}. This verifies Lemma 2.

Step 2 : *Construction of the families* $\{V_t\}$ *and* $\{W_t\}$. We employ the second part of the construction due to Duval [12].

Fix an orientation on the real hypersurface $\partial\Omega'$ and the disc X. This allows one to define an orientation on the leaves of the characteristic foliation. Let $p \in X \setminus \{0\}$ and v_1 and v_2 be vectors in the tangent space T_pX giving a positively oriented basis there. A nonzero vector v tangent to the leaf of the characteristic foliation through p defines the positive orientation on this leaf if the triple v_1, v_2, Jv is a positively oriented basis of $T_p(\partial\Omega')$. Here J denotes the standard complex structure of $\mathbb{C}^2$, that is, the vector Jv can be identified with iv.

We argue by contradiction. Let $p \in X \setminus \{0\}$ be a totally real point such that p lies in the leaf γ of the characteristic foliation, $p \in X^{\mathrm{ess}}$, but X^{ess} does not meet both sides of γ. Fix an open neighborhood U' of p small enough so that 0 does not lie in $\bar{U}'$ and $\Omega' \cap U'$ is biholomorphic to a strictly convex domain. More precisely, one can assume that there are local coordinates (z', w') in U' such that p corresponds to the origin $0'$, U' is a ball, and $\Omega' \cap U'$ is strictly convex. Let x and y be points on X near $0'$ that lie on the same leaf of the characteristic foliation. Assume that the direction from x to y along this leaf is positive for the orientation described earlier. Denote by $L(x, y)$ the complex line through x and y. Then, $L(x, y)$ meets $X \cap U'$ at the points x and y only; this intersection is transversal, positive at x, and negative at y; see [12, Lemma 2]. Denote by $\Delta(x, y)$ the intersection of the line $L(x, y)$ with the ball U'.

Denote by γ' a leaf of the characteristic foliation near p parallel to γ. By assumption, one can choose γ' to be disjoint from X^{ess} in U'. Consider a (short) arc $\alpha : [0, 1] \to X \cap U'$ such that $\alpha(0) = p$, $\alpha(1) = p'$, where p' is a point of γ' and such that for $t > 0$ the point $\alpha(t)$ is on the same side of γ as the leaf γ'. Finally, choose a point $x \in \gamma$ that precedes p, and a corresponding point $x' \in \gamma'$ that precedes p'. Let $\beta : [0, 1] \to X$ be an arc in γ' with $\beta(0) = x'$, $\beta(1) = p'$.

Now we are able to construct the first family $\{V_t\}$ of analytic curves. We begin with the family $\Delta(x', \alpha(t))$, where $0 \le t \le 1$. As it was already mentioned, the line $L(x, p)$ intersects X with positive sign at p. This property is stable with respect to continuous deformations of complex lines $L(q, p)$ where q moves from x to x' in X. Hence, the first disc $V_0 = \Delta(x', \alpha(0))$ of our family intersects X at p with positive sign. We continue this family with the discs $\Delta(\beta(t), p')$, $0 \le t \le 1$, starting with $t = 0$. When $t = 1$ we arrive to the complex tangent $\Delta(p', p')$. The final piece of the family $\{V_t\}$ is obtained by the translation $\Delta(p', p')$ into the complement of Ω' along the outward normal direction to $\partial\Omega'$ at p'. Similarly, we proceed with the construction of the second family $\{W_t\}$ using a point $y \in \gamma$ that succeeds p along γ and a corresponding point $y' \in \gamma'$ that succeeds p' along γ'.

The curves V_0 and W_0 meet transversally at p with opposite signs of intersection and for $t > 0$ the curves V_t and W_t do not meet X^{tr}. In the aforementioned local coordinates (z', w') on U' these curves are intersections of the complex lines with U' described earlier, that is, the corresponding functions f_t, g_t are degree one polynomials in (z', w'). Since the families $\{V_t\}$ and $\{W_t\}$ can be chosen arbitrarily close to the complex tangent line to $\partial\Omega'$ at p, their boundaries are contained in $\partial U'$ and do not intersect $\bar{\Omega}'$. Therefore V_t and W_t are analytic curves in a suitably chosen global neighborhood O of $\overline{\Omega' \cap \mathbb{B}(0, \varepsilon)}$ in $\mathbb{C}^2$. Now Step 1 can be used. Lemma 2 implies that p does not lie in X^{ess}, which gives a contradiction. Proposition 2 is proved.

3.3 Proof of the main results

We now prove the main results of the paper assuming that Proposition 1 holds.

Proof of Theorem 1. Let γ_1 and γ_2 be as in Proposition 1. It follows from Propositions 1 and 2 that X^{tr} is contained in the union $\gamma_1 \cup \gamma_2$, and Rossi's maximum principle implies $X^{\text{ess}} \subset \widehat{\gamma_1 \cup \gamma_2}$.

A rectifiable arc is polynomially convex [29]. Moreover, if Y is compact and polynomially convex, and Γ is a compact connected set of finite length, then the set

$(\widehat{Y \cup \Gamma}) \setminus (Y \cup \Gamma)$ is either empty or contains a complex purely one-dimensional analytic subvariety of $\mathbb{C}^2 \setminus (Y \cup \Gamma)$ (see [31, p. 122]). By taking Y and Γ to be our rectifiable curves γ_j, we see as in the proof of Corollary 2 that their union cannot bound a complex one-dimensional variety. Therefore, $\gamma_1 \cup \gamma_2$ is polynomially convex: $\widehat{\gamma_1 \cup \gamma_2} = \gamma_1 \cup \gamma_2 \subset X$. As a consequence we obtain that X^{ess} also is contained in X. Let p be a point of $\hat{X} \setminus X$. Then $p \in X^{\text{ess}} \setminus X$ which is impossible. This implies that $\hat{X} \setminus X$ is empty. Hence, X is polynomially convex. Theorem 1 is proved. ■

Proof of Corollary 1. Let $\phi(0) = p$. By Theorem 1 there exists $\varepsilon > 0$ such that $X = \overline{\phi(\Sigma) \cap \mathbb{B}(p, \varepsilon)}$ is polynomially convex. We may further assume that $\phi(\Sigma) \cap \partial\mathbb{B}(p, \varepsilon)$ is a rectifiable and even smooth curve. By the result of Anderson et al. [3, Theorem 1.5], if X is a polynomially convex compact subset of $\mathbb{C}^n$, and X_0 is a compact subset of X such that $X \setminus X_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1, then continuous functions on X can be approximated by polynomials if and only if this can be done on X_0. We apply this result to $X = \overline{\phi(\Sigma) \cap \mathbb{B}(p, \varepsilon)}$ and $X_0 = \{p\} \cup (\phi(\Sigma) \cap \partial\mathbb{B}(p, \varepsilon))$. The set X_0, is polynomially convex. Indeed, if not, we obtain as in the proof of Theorem 1 that $\hat{X}_0 \setminus X_0$ contains a complex purely one-dimensional analytic subvariety V of $\mathbb{C}^2 \setminus X_0$. But then V is contained in $\hat{X}$, which contradicts Theorem 1. Furthermore, by [30] or [31, p. 122], continuous functions on X_0 can be approximated by polynomials. From this the corollary follows. ■

The rest of the paper is devoted to the proof of Proposition 1.

4 Reduction to a Dynamical System

In this section, we deduce the dynamical systems describing the pull-back in $\mathbb{R}^2_{(t,s)}$ of the characteristic foliations on Σ and Σ'. In Sections 6 and 7, we will discuss the topological behavior of these foliations near the origin. For simplicity, the integral curves of these dynamical systems will also be called the leaves of the characteristic foliation.

4.1 Foliation on Σ

The tangent plane to $\Sigma \setminus \{0\}$ is spanned by the vectors

$$X_t = \begin{pmatrix} s \\ 2t^2 \\ 2t \\ 0 \end{pmatrix}, \quad X_s = \begin{pmatrix} t \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The directional vector of the characteristic line field is determined from the equation

$$X = \alpha X_t + \beta X_s, \tag{12}$$

where $\alpha = \alpha(t,s)$ and $\beta = \beta(t,s)$ are some smooth functions on $\mathbb{R}^2 \setminus \{0\}$, and the vector X belongs to the complex tangent $H_{\pi(t,s)}M$. Let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

Multiplication by i of a vector in $\mathbb{C}^2$ corresponds to multiplication by J of the corresponding vector in $\mathbb{R}^4$. For $v \in T_pM$, the inclusion $v \in H_pM$ holds if and only if $v, iv \in T_pM$. Therefore,

$$X \in H_{\pi(t,s)}M \iff \langle J(\alpha X_t + \beta X_s), \nabla\rho\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean product in $\mathbb{R}^4$, and $\nabla\rho$ is the gradient of the function ρ. Therefore, we can choose

$$\alpha = \langle JX_s, \nabla\rho\rangle, \quad \beta = -\langle JX_t, \nabla\rho\rangle. \tag{13}$$

A calculation yields

$$\nabla\rho = (2ts, 3t^3, -s^2 - 3t^4, -2t^2 s),$$

and

$$\alpha = -3t^3 - ts^2 - 3t^5,$$
$$\beta = s^3 + 4t^2 s + 7st^4.$$

Thus,

$$X = \alpha X_t + \beta X_s = \alpha\, \mathrm{d}\pi \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\, \mathrm{d}\pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{d}\pi \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{14}$$

where $\mathrm{d}\pi$ is the differential of the map π. It follows that the characteristic foliation on $\Sigma \setminus \{0\}$ (or, more precisely, its pull-back on $\mathbb{R}^2\setminus\{0\}$ by the parametrization map π) is given by the system of ODEs of the form

$$\dot{t} = -3t^3 - ts^2 - 3t^5, \qquad (15)$$
$$\dot{s} = s^3 + 4t^2 s + 7st^4,$$

where the dot denotes the derivative with respect to the time variable τ.

4.2 Foliation on Σ'

Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$f := \psi \circ \phi \circ \pi,$$

where we use the notation of the previous section. The directional vector of the characteristic foliation on Σ' is determined by

$$X' = \alpha X'_t + \beta X'_s,$$

where $X'_t = \partial f/\partial t$ and $X'_s = \partial f/\partial s$ and $\alpha = \alpha(t,s)$, and $\beta = \beta(t,s)$ are some smooth functions on $\mathbb{R}^2 \setminus \{0\}$ that are chosen in such a way that vector X' belongs to the complex tangent $H_{f(t,s)}M'$. We have

$$X' \in H_{f(t,s)}M' \iff \langle J(\alpha X'_t + \beta X'_s), \nabla\rho' \rangle = 0,$$

where ρ' is a defining function of M', and the gradient $\nabla\rho'$ is expressed in terms of (t,s) using the parametrization f. Therefore, we can choose

$$\alpha(t,s) = \langle JX'_s, \nabla\rho' \rangle, \quad \beta(t,s) = -\langle JX_t, \nabla\rho' \rangle. \qquad (16)$$

Thus,

$$X' = \alpha X'_t + \beta X'_s = df \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \qquad (17)$$

It follows that the characteristic foliation on Σ' is determined by the system of ODEs of the form

$$\dot{t} = \alpha(t,s), \qquad (18)$$
$$\dot{s} = \beta(t,s).$$

We write $f(t,s) = (f_1(t,s), \ldots, f_4(t,s))$, where using (6) and (1) we may express each f_j as a power series in (t,s):

$$\begin{aligned} f_1(t,s) &= x + \sum_{j+k+l+m\geq 2} \tilde{f}^1_{jklm}\, x^j u^k y^l v^m \\ &= ts + f^1_{02}s^2 + f^1_{12}ts^2 + f^1_{21}t^2 s + f^1_{03}s^3 + \sum_{j+k\geq 4} f^1_{jk}t^j s^k, \end{aligned} \tag{19}$$

where $\tilde{f}^1_{jklm}$ and f^1_{jk} are real numbers. Similarly,

$$\begin{aligned} f_2(t,s) &= u + \sum_{j+k+l+m\geq 2} \tilde{f}^2_{jklm}\, x^j u^k y^l v^m \\ &= \frac{2}{3}t^3 + f^2_{02}s^2 + f^2_{12}ts^2 + f^2_{21}t^2 s + f^2_{03}s^3 + \sum_{j+k\geq 4} f^2_{jk}t^j s^k. \end{aligned} \tag{20}$$

Denote by e_{jk} the entries of the matrix E in (6). Then

$$\begin{aligned} f_3(t,s) &= e_{11}x + e_{12}u + g_{11}y + g_{12}v + \sum_{j+k+l+m\geq 2} \tilde{f}^3_{jklm}\, x^j u^k y^l v^m \\ &= g_{12}s + g_{11}t^2 + e_{11}ts + f^3_{02}s^2 + \frac{2e_{12}}{3}t^3 + f^3_{12}ts^2 + f^3_{21}t^2 s + f^3_{03}s^3 + \sum_{j+k\geq 4} f^3_{jk}t^j s^k; \qquad (21) \\ f_4(t,s) &= e_{21}x + e_{22}u + g_{12}y + g_{22}v + \sum_{j+k+l+m\geq 2} \tilde{f}^4_{jklm}\, x^j u^k y^l v^m \\ &= g_{22}s + g_{12}t^2 + e_{21}ts + f^4_{02}s^2 + \frac{2e_{22}}{3}t^3 + f^4_{12}ts^2 + f^4_{21}t^2 s + f^4_{03}s^3 + \sum_{j+k\geq 4} f^4_{jk}t^j s^k. \qquad (22) \end{aligned}$$

From these formulas we immediately obtain

$$X'_t = \begin{pmatrix} s + 2f^1_{21}ts + f^1_{12}s^2 \\ 2t^2 + 2f^2_{21}ts + f^2_{12}s^2 \\ 2g_{11}t + e_{11}s + 2e_{12}t^2 + 2f^3_{21}ts + f^3_{12}s^2 \\ 2g_{12}t + e_{21}s + 2e_{22}t^2 + 2f^4_{21}ts + f^4_{12}s^2 \end{pmatrix} + o(|(t,s)|^2), \tag{23}$$

and

$$X'_s = \begin{pmatrix} t + 2f^1_{02}s + f^1_{21}t^2 + 2f^1_{12}ts + 3f^1_{03}s^2 \\ 2f^2_{02}s + f^2_{21}t^2 + 2f^2_{12}ts + 3f^2_{03}s^2 \\ g_{12} + e_{11}t + 2f^3_{02}s + f^3_{21}t^2 + 2f^3_{12}ts + 3f^3_{03}s^2 \\ g_{22} + e_{21}t + 2f^4_{02}s + f^4_{21}t^2 + 2f^4_{12}ts + 3f^4_{03}s^2 \end{pmatrix} + o(|(t,s)|^2). \tag{24}$$

The defining equation of M' can be chosen to be $\rho \circ (\psi \circ \phi)^{-1}$, where ρ defines M as in (2). Let (x', u', y', v') be the coordinates in the target domain of $\psi \circ \phi$, in particular, we have $x' = f_1$, $u' = f_2$, $y' = f_3$, and $v' = f_4$. Let

$$(D(\psi \circ \phi)(0))^{-1} = \begin{pmatrix} I_2 & 0 \\ E' & G' \end{pmatrix}, \quad E' = (e'_{jk}), \quad G' = (g'_{jk}). \tag{25}$$

Then

$$\begin{aligned}(\psi \circ \phi)^{-1}(x', u', y', v') = \Bigg(& x' + \sum_{j+k+l+m \geq 2} h^1_{jklm}\, x'^j u'^k y'^l v'^m,\ u' + \sum_{j+k+l+m \geq 2} h^2_{jklm}\, x'^j u'^k y'^l v'^m, \\ & e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sum_{j+k+l+m \geq 2} h^3_{jklm}\, x'^j u'^k y'^l v'^m, \\ & e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sum_{j+k+l+m \geq 2} h^4_{jklm}\, x'^j u'^k y'^l v'^m \Bigg). \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} & \rho'(x', u', y', v') \\ & = \Bigg(x' + \sum_{j+k+l+m \geq 2} h^1_{jklm}\, x'^j u'^k y'^l v'^m \Bigg)^2 - \Big(e'_{11}x' + \cdots + g'_{12}v' + \sum h^3_{jklm}\, x'^j u'^k y'^l v'^m \Big) \\ & \quad \cdot \Big(e'_{21}x' + \cdots + g'_{22}v' + \sum h^4_{jklm}\, x'^j u'^k y'^l v'^m \Big)^2 + \frac{9}{4} \Bigg(u' + \sum_{j+k+l+m \geq 2} h^2_{jklm}\, x'^j u'^k y'^l v'^m \Bigg)^2 \\ & \quad - \Big(e'_{11}x' + \cdots + g'_{12}v' + \sum h^3_{jklm}\, x'^j u'^k y'^l v'^m \Big)^3. \end{aligned} \tag{27}$$

Note that in (27) the only quadratic terms are x'^2 and $\frac{9}{4}u'^2$. By taking partial derivatives in this expression with respect to x', u', y', and v', and expressing the resulting vector in terms of (t, s) we will obtain the coordinates of the vector

$$\nabla\rho' = \left(\frac{\partial\rho'}{\partial x'}, \frac{\partial\rho'}{\partial u'}, \frac{\partial\rho'}{\partial y'}, \frac{\partial\rho'}{\partial v'}\right) = (R_x(t,s), R_u(t,s), R_y(t,s), R_v(t,s)).$$

To determine the phase portrait of the characteristic foliation, we will only need some low-order terms in the power series

$$\alpha(t,s) = \sum_{j,k\geq 0} \alpha_{jk} t^k s^j, \quad \beta(t,s) = \sum_{j,k\geq 0} \beta_{jk} t^k s^j.$$

Therefore, instead of explicit differentiation of (27), we will employ a different strategy for computing coefficients of the terms of lower degree in the (t, s)-Taylor expansion of α and β.

4.3 The power series of α

We have

$$\alpha(t,s) = \langle JX'_s, \nabla\rho'\rangle = -(X'_s)_3 \cdot R_x - (X'_s)_4 \cdot R_u + (X'_s)_1 \cdot R_y + (X'_s)_2 \cdot R_v. \tag{28}$$

We proceed in several steps computing the coefficients in the expansion for α. To begin with, there cannot be a free term in the power series of α because every term in $\nabla\rho'$ will necessarily have positive degree in t or s.

Term t: Since no component of $\nabla\rho'$ can contain a degree-zero term or the monomial t, there is no term t in α.

Term s: The first two components of X'_s do not contain free terms; therefore, monomial s can appear in α only if R_x or R_u will contain it. By inspection of (19)–(22), we see that y' and v' are the only terms that can produce monomial s. Therefore, for s to appear in R_x or R_u, the function ρ' must contain at least one of the terms $x'y'$, $x'v'$, $u'y'$, or $u'v'$. However, from (27) neither of these terms exists. Thus, there is no monomial s in the power series of α.

Term ts: We inspect terms in X'_s of degree lower than ts. These appear in $(X'_s)_1$ (terms t and s), in $(X'_s)_2$ (term s), in $(X'_s)_3$ (a free term, t and s), and in $(X'_s)_4$ (a free term, t and s). Therefore, for ts to appear in α, at least one of the following options must occur:

(1) either R_x or R_u has t, s or ts;
(2) R_y has either t or s;

(3) R_v has t.

Of these three options only (1) can happen: ρ' contains the term x'^2, and therefore, R_x contains $2ts$. It follows now from (19), (24), and (28) that $\alpha_{11} = -2g_{12}$.

To simplify further considerations, we note that term t cannot occur in any of the components of the vector $\nabla\rho'$.

Term t^2: By inspection of X'_s, we conclude that either R_x or R_u has term t^2, so $\nabla\rho'$ must have either $x'y'$, $x'v'$, $u'y'$, or $u'v'$, neither of which appears. This means that α does not contain term t^2.

Term s^2: By inspection of X'_s, the following options are possible:

(1) either R_x or R_u has s or s^2;

(2) either R_y or R_v has term s.

Option (2) is impossible, but ρ' can have terms u'^2, $u'v'^2$, or $u'y'^2$, which gives (1). We have the following expression for α_{02}, which depends on the coefficients of the Taylor expansion for $(\psi \circ \phi)^{-1}$:

$$\alpha_{02} = \frac{9}{2}(h_{0002}^2 g_{22}^2 + f_{02}^2 + h_{0020}^2 g_{12}^2).$$

Term t^3: By analyzing of X'_s, the following options are possible:

(1) either R_x or R_u has at least one of t^2 or t^3;

(2) R_y has t^2.

Option (2) can happen only if ρ' would have y'^2 or $y'v'$, which is impossible. For the same reason in option (1) terms R_x or R_u cannot produce t^2. The only term in $\nabla\rho'$ that can produce t^3 is u'. Therefore, the only possibility in (1) is the term t^3 in R_u, which indeed happens since ρ' contains u'^2. It follows that $\alpha_{30} = -3g_{22}$.

Thus,

$$\alpha(t, s) = -2g_{12}ts + \alpha_{02}s^2 - 3g_{22}t^3 + \sum_{j+k>2,(j,k)\neq(3,0)} \alpha_{jk}t^j s^k.$$

4.4 The power series of β

We have

$$\beta(t, s) = -\langle JX'_t, \nabla\rho'\rangle = (X'_t)_3 \cdot R_x + (X'_t)_4 \cdot R_u - (X'_t)_1 \cdot R_y - (X'_t)_2 \cdot R_v.$$

Again, there cannot be a free term in β because every term in $\nabla\rho'$ will necessarily have positive degree in t or s. Further, no component in $\nabla\rho'$ can produce a term t, and so the power series of β cannot contain monomial t.

Term s: Since no component of X_t' contains a free term, β cannot have monomial s.

Term ts: By inspection of X_t', we conclude that either R_x or R_u must have term s, which is impossible. Hence, β does not contain monomial ts.

Terms t^2 and s^2: Analogous considerations show that these terms cannot appear in β.

Term t^2s: By inspection of X_t' the following is possible for R:

(1) R_x has at least one of t^2, s, or ts;
(2) R_u has at least one of t^2, s, or ts;
(3) R_y has t^2;
(4) R_v has s.

Options (3) and (4) imply that ρ' has v'^2, y'^2, or $v'y'$, neither of which is possible. Option (2) implies that ρ' has $u'y'$, $u'v'$, and $u'x'$. Neither of these terms is present in ρ', so (2) is also not possible. Option (1) implies that ρ' has at least one of $x'y'$, $x'v'$, and x'^2. Only the latter happens, and so $\beta_{21} = 4g_{11}$.

Term ts^2: This term can appear in β. We have

$$\beta_{12} = 2e_{11} + 6g_{12} f_{02}^2.$$

Term t^3: By analyzing of X_t', the only option is that either R_x or R_u has term t^2. This is however not possible.

Term t^4: The possibilities for R are as follows:

(1) R_x has at least one of t^2 or t^3;
(2) R_u has at least one of t^2, or t^3;
(3) R_v has t^2.

Option (3) cannot occur. The only possible option in (1) or (2) is that t^3 appears in R_u. This comes from the term u'^2 in ρ'. It follows that $\beta_{04} = 6g_{12}$.

Term s^3: We have

$$\beta_{03} = 2e_{11} f_{02}^1 + \tfrac{9}{2} e_{21} f_{02}^2.$$

Combining everything together we get

$$\beta(t,s) = 4g_{11}t^2s + \beta_{12}ts^2 + \beta_{03}s^3 + 6g_{12}t^4 + \sum_{j+k>3,\ (j,k)\neq(4,0)} \beta_{jk}t^js^k.$$

We note that if ϕ is merely a smooth diffeomorphism, then these calculations give the values for the jets of α and β at the origin of the corresponding orders. In either case, the characteristic foliation on Σ' is given by

$$\begin{aligned} \dot t &= \alpha(t,s) = -2g_{12}ts + \alpha_{02}s^2 - 3g_{22}t^3 + o(|t|^3 + |s|^2 + |ts|), \\ \dot s &= \beta(t,s) = 4g_{11}t^2s + \beta_{12}ts^2 + \beta_{03}s^3 + 6g_{12}t^4 + o(|t^2s| + |ts^2| + |s|^3 + |t|^4). \end{aligned} \tag{29}$$

It is easy to see that for a generic symplectomorphism $\phi : (x,u,y,v) \mapsto (x',u',y',v')$ and a generic ψ the coefficients α_{02}, β_{12}, and β_{03} do not vanish. Indeed, if ψ is close to the identity map and the component u' of ϕ contains the term av^2 with $a \neq 0$, then $f^2_{02} \neq 0$ and α_{02}, β_{12}, and β_{03} do not vanish. Therefore, they do not vanish generically.

Remark. It follows from these considerations that our restriction on ϕ to be generic involves only the 2-jet of ϕ at the origin. In other words, it suffices to require in Theorem 1 that ϕ has a generic 2-jet at the origin. □

Lemma 3. Let ϕ be a local symplectomorphism near the origin, and let $\mathcal{X}$ be the vector field near the origin in $\mathbb{R}^2$ corresponding to the characteristic foliation on Σ'. Then $\mathcal{X}$ does not vanish outside the origin. □

Proof. Since ϕ is symplectic, $\phi(\Sigma \setminus \{0\})$ is a Lagrangian surface, in particular, totally real. Therefore, $\psi \circ \phi(\Sigma \setminus \{0\})$ does not contain complex points. Further, it easily follows from (16) that $\alpha(t_0,s_0) = \beta(t_0,s_0) = 0$, $(t_0,s_0) \neq 0$ if and only if $f(t_0,s_0)$ is a complex point of Σ'. From this the result follows. ■

5 Generalities on Planar Vector Fields

For the proof of Proposition 1, we need to determine the topological structure of the *orbits* or *maximal integral curves* associated with the vector fields defined by (15) and (29). Both systems have higher order degeneracy (the linear part vanishes) at the origin, and consequently, it is a *nonelementary* singularity of (15) and (29). Therefore, standard results, such as the Hartman–Grobman theorem, do not apply here. Instead, we will use some more advanced tools from dynamical systems. We will be primarily interested in understanding the topological picture of (15) and (29) near the origin up to a homeomorphism preserving the orbits. In this section, we outline relevant results and recall some common terminology.

5.1 Finite jet determination of the phase portrait

The local phase portrait of a vector field near a nonelementary isolated singularity can be determined through a finite sectorial decomposition. This means that a neighborhood of the singularity is divided into a finite number of sectors with certain orbit behavior in each sector. If the vector field has at least one *characteristic* orbit (i.e., orbits approaching in positive or negative time the singularity with a well-defined slope limit), then the boundaries of the sectors can be chosen to be characteristic orbits. The overall portrait is then understood by gluing together the topological picture in each sector. The general result due to Dumortier [8] (see also [9]) can be stated as follows:

Suppose that a C^∞-smooth vector field $\mathcal{X}$ singular at the origin in $\mathbb{R}^2$ satisfies the Łojasiewicz inequality

$$|\mathcal{X}(x)| \geq c|x|^k, \quad c > 0, \ k \in \mathbb{N},$$

for $x \in \mathbb{R}^2$ is some neighborhood of the origin. Then $\mathcal{X}$ has the finite sectorial decomposition property, that is, the origin is either a center (all orbits are periodic), a focus/node (all orbits terminate at the origin in positive or negative time), or there exists a finite number of characteristic orbits that bound sectors with a well-defined orbit behavior (hyperbolic, parabolic, or elliptic). If the vector field $\mathcal{X}$ has a characteristic orbit, then its phase portrait is determined by its jet of finite order k, in the sense that any other vector field with the same jet of order k at the origin has the phase portrait homeomorphic to that of $\mathcal{X}$. Further, whether the vector field $\mathcal{X}$ has a characteristic orbit depends only on a jet of $\mathcal{X}$ of some finite order.

The original proof of this result in [8] is based on the desingularization by means of successive (homogeneous) blow-ups. After each blow-up, the singularity is replaced by a circle, and after a finite number of such blow-ups, one obtains a vector field with only nondegenerate singularities. The construction of the blow-up maps depends only on a finite order jet of the original vector field at the origin. From the configuration of the singularities of the modified system on the preimage of the origin under the composition of blow-ups, it is always possible to deduce whether the original vector field has a characteristic orbit. If such an orbit exists, then the singularity is not a center or a focus, and the phase portrait is determined by a jet of finite order. Further, the Łojasiewicz inequality holds for any real analytic vector field in a neighborhood of an isolated singularity (see, e.g., [4]) and, in particular, in our case, in view of Lemma 3.

Alternatively, it is possible to use *quasihomogeneous* blow-ups, which are chosen according to the Newton diagram associated with $\mathcal{X}$ (see [26]). The advantage is that this gives a computational algorithm for constructing the sectorial decomposition for a particular system. A detailed discussion of this approach for real analytic systems is given in Bruno [6] in the language of normal forms. Using Bruno's method, we will show that for a real analytic ϕ in general position, the vector field defined by (29) will always have a characteristic orbit, and its phase portrait near the origin is a saddle.

If in Theorem 1 the map ϕ is smooth, then the vector field corresponding to the characteristic foliation is only smooth, and the Łojasiewicz inequality imposes additional assumption on the vector field, and therefore on ϕ. The Łojasiewicz condition depends on the jet of the vector field at the origin and holds for all jets outside a set of infinite codimension in the space of jets, but it is not clear whether for a generic smooth symplectomorphism the inequality is satisfied. However, assuming that the Łojasiewicz condition does hold, the topological picture of the characteristic foliation is determined by its finite jet at the origin. Therefore, we may consider a polynomial vector field obtained by truncation of (29) at sufficiently high order without distorting the phase portrait of the system. After that we may apply Bruno's method to determine its geometry. Thus, in Theorem 1 we may assume that ϕ is a generic smooth symplectomorphism such that the vector field corresponding to the characteristic foliation satisfies the Łojasiewicz inequality.

If in Theorem 1 the map ϕ is a real analytic diffeomorphism with $D\phi(0)$ symplectic, then all of the arguments go through provided that the vector field (29) vanishes at the origin only. The latter holds for the following reason: consider near the origin the complexification F of the real analytic map $f = \psi \circ \phi \circ \pi : \mathbb{R}^2 \to \mathbb{C}^2$. Then $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic map such that $F|_{\mathbb{R}^2_{(t,s)}} = f$, in particular, $F(\mathbb{R}^2) = \Sigma'$. Moreover, since f has rank 2 outside the origin, it follows that the Jacobian of F does not vanish on $\mathbb{R}^2 \setminus \{0\}$, and therefore, F is a local biholomorphism near any point on $\mathbb{R}^2 \setminus \{0\}$. But this implies that $\Sigma' \setminus \{0\}$ is totally real, and therefore the characteristic foliation has no singularities outside the origin. Thus, Theorem 1 holds under the assumption that ϕ is a generic real analytic diffeomorphism with $D\phi(0)$ symplectic.

In the remaining part of this section, we outline general theory of normal forms and sector decomposition of dynamical systems due to Bruno [6], while the actual numerical calculations for (15) and (29) are presented in Section 6.

5.2 Normal forms for elementary singularities

We state three theorems due to Bruno on normal forms for vector fields near an isolated *elementary* singularity. Consider the system

$$\dot{x}_i = \lambda_i x_i + \sigma_i x_{i-1} + \varphi_i(X), \quad i = 1, 2, \tag{30}$$

where x_i are smooth functions of a real variable and $X = (x_1, x_2)$. Here σ_j, λ_j are real, $\sigma_1 = 0$ and the series φ_i does not contain constant or linear terms. In other words, using the notation $X^Q = x_1^{q_1} x_2^{q_2}$ for $Q = (q_1, q_2) \in \mathbb{Z}^2$, we can write

$$\varphi_i(X) = \sum_Q f_{iQ} X^Q, \quad i = 1, 2,$$

where $q_j \geq 0$, $q_1 + q_2 > 0$. The main assumption is that at least one of the eigenvalues λ_i is nonzero that is $|\lambda_1| + |\lambda_2| \neq 0$. This means that the origin is an elementary singularity. We suppose below that all systems considered in the Normal Forms Theorems are real analytic, though the considerations in the formal power series category also make sense.

The goal is to transform system (30) to the simplest possible form

$$\dot{y}_i = \tilde{\psi}_i(Y) := \lambda_i y_i + \sigma_i y_{i-1} + \psi_i(Y), \quad i = 1, 2 \tag{31}$$

by a local invertible change of coordinates

$$x_i = y_i + \xi_i(Y), \quad i = 1, 2, \tag{32}$$

where the series ξ_i in $Y = (y_1, y_2)$ do not contain constant or linear terms:

$$\xi_i(Y) = \sum_{|Q|>1} h_{iQ} Y^Q, \quad i = 1, 2.$$

We use the notation $|Q| = |q_1| + |q_2|$. Such a change of coordinates in general is not real analytic, that is, the series ξ_i can be divergent. For this reason we consider formal power series ξ_i and refer to (32) as a *formal* changes of coordinates.

It is convenient to use the representation

$$\tilde{\psi}_i(Y) = y_i g_i(Y) = y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \quad i = 1, 2, \tag{33}$$

where

$$N_1 = \{Q = (q_1, q_2) \in \mathbb{Z}^2 : q_1 \geq -1, q_2 \geq 0, q_1 + q_2 \geq 0\},$$

$$N_2 = \{Q = (q_1, q_2) \in \mathbb{Z}^2 : q_1 \geq 0, q_2 \geq -1, q_1 + q_2 \geq 0\}.$$

Set $\Lambda = (\lambda_1, \lambda_2)$ and denote by $\langle \bullet, \bullet \rangle$ the standard inner product in $\mathbb{R}^2$.

Principal normal form [6, Chapter II, Section 1, Theorem 2, p. 105]: There exists a formal change of coordinates (32) such that system (30) in the new coordinates takes the form (31), where $g_{iQ} = 0$ for $Q = (q_1, q_2)$ satisfying $\langle Q, \Lambda \rangle = q_1\lambda_1 + q_2\lambda_2 \neq 0$.

Therefore, the normal form (31) contains only terms of the form $y_i g_{iQ} Y^Q$ satisfying

$$\langle Q, \Lambda \rangle = 0. \tag{34}$$

Such terms are called *resonant*.

The fundamental question on the convergence of a normalizing change of coordinates for an analytic system (30) is discussed in [6]. In the cases that we will consider in this article, normalizing changes of coordinates (32) will be analytic or at least C^∞-smooth local diffeomorphisms (see [6]). This is sufficient for the study of local topological behavior of integral curves.

Consider now a more general system of two differential equations in two variables of the form

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{Q \in \mathbf{V}} f_{iQ} X^Q = \lambda_i x_i + x_i f_i, \quad i = 1, 2, \tag{35}$$

where $\Lambda = (\lambda_1, \lambda_2) \neq 0$. The set $\mathbf{V} \subset \mathbb{Z}^2$, over which the exponents Q run, is to be prescribed. In the hypothesis of the Principal Normal Form Theorem, $\varphi_i(X)$ are power series in nonnegative powers of variables and the corresponding $\mathbf{V}$ is almost completely contained in the first quadrant of the plane.

To formulate a weaker assumption on $\mathbf{V}$, we consider two vectors R^* and R_* in $\mathbb{R}^2$ contained in the second and the forth quadrant, respectively, and denote by $\mathbf{V}$ the sector bounded by R^* and R_* and containing the first quadrant. We assume that R^* and R_* are such that $\mathbf{V}$ has an angle less than π. As a consequence, the sector $\mathbf{V}$ is the convex cone generated by R^* and R_*, that is, consists of the vectors $\alpha_1 R^* + \alpha_2 R_*$ with $\alpha_j \geq 0$. We use the notation $|X| = (|x_1|, |x_2|)$ and $|X|^Q = |x_1|^{q_1}|x_2|^{q_2}$.

Denote by $\mathcal{V}(X)$ the space of power series $\sum_Q f_Q X^Q$, where $Q \in \mathbf{V}$. Since in our situation such a series can have an infinite number of terms with negative exponents (even after multiplication by x_i), the notion of its convergence requires clarification. Consider first a numerical series

$$\sum_{Q\in\mathbb{Z}^2} a_Q, \tag{36}$$

where the indices Q run through $\mathbb{Z}^2$. Let (Ω_n) be an increasing exhausting sequence of bounded domains in $\mathbb{R}^2$. Set

$$S_n = \sum_{Q\in\Omega_n} a_Q$$

(the partial sums). If the sequence (S_n) admits the limit S and this limit is independent of the choice of the sequence (Ω_n), then we say that series (36) converges to the sum S. It is well-known that if for some sequence (Ω_n) the sequence of the partial sums of the series

$$\sum_{Q\in\mathbb{Z}^2} |a_Q| \tag{37}$$

converges, then series (36) and (37) converge. In this case we say that series (36) converges absolutely.

Under the aforementioned assumptions on R^* and R_* a series of class $\mathcal{V}(X)$ is called *convergent* if it converges absolutely in the set

$$\mathcal{U}_{\mathbf{V}}(\varepsilon) = \{X : |X|^{R_*} \leq \varepsilon, |X|^{R^*} \leq \varepsilon, |x_1| \leq \varepsilon, |x_2| \leq \varepsilon\}, \tag{38}$$

for some $\varepsilon > 0$. As explained in detail in [6], this subset of the real plane is a natural domain of convergence for such a series. As an example we notice that when the sector $\mathbf{V}$ is defined by the vectors $R_* = (1, 0)$ and $R^* = (0, 1)$, that is, coincides with the first quadrant, then the class $\mathcal{V}(X)$ coincides with the class of usual power series with non-negative exponents and the set $\mathcal{U}_{\mathbf{V}}(\varepsilon)$ coincides with the bidisc of radius ε.

Let **V** be a sector that determines system (35). We consider changes of variables of the form

$$x_i = y_i + y_i h_i(Y), \quad i = 1, 2, \tag{39}$$

where $h_i \in \mathcal{V}(Y)$, that is, $h_i(Y) = \sum_{Q \in \mathbf{V}} h_{iQ} Y^Q$. In the new coordinates the system takes the form

$$\dot{y}_i = \lambda_i y_i + y_i g_i(Y), \quad i = 1, 2. \tag{40}$$

Second normal form [6, Chapter II, Section 2, Theorem 1, p. 128]: Suppose that **V** is a sector as described earlier. Then system (35) can be transformed by a formal change of variables (39) into a normal form (40) with $g_i \in \mathcal{V}(Y)$. The coefficients of g_i satisfy $g_{iQ} = 0$ if $\langle Q, \Lambda \rangle \neq 0$.

The normalizing change of coordinates in this theorem in general is not convergent, even if system (35) is analytic. However, such a change of coordinates is always convergent or C^∞-smooth in $\mathcal{U}_{\mathbf{V}}(\varepsilon)$. For this reason the behaviors of the integral curves of systems (35) and (40) coincide in the sector given by (38) for sufficiently small $\varepsilon > 0$.

The third theorem deals with the case somewhat intermediate with respect to the two previous theorems. Let **V** be the sector in (35) as defined already by the vectors R^* and R_*. Assume that $R^* = (r_1^*, r_2^*)$ and $R_* = (r_{1*}, -1)$ with $r_1^* < 0 < r_2^*$, $r_{1*} > 0$, and $|r_1^*/r_2^*| < r_{1*}$. Note that the conditions on r_1^*, r_2^*, and r_{1*} exactly mean that R^* and R_* are in the second and forth quadrants, respectively, and the angle of **V** is less than π.

The additional assumption that we impose is that the expressions on the right-hand side of (35) are the series in integer nonnegative powers of x_2. Since the series $f_1(X)$ does not contain negative powers of x_2, the coefficient f_{1Q} in $f_1(X)$ vanishes unless the vector Q lies in the sector

$${}_1\mathbf{V} = \{Q : Q = \alpha_1 R^* + \alpha_2 \cdot (1, 0),\ \alpha_1, \alpha_2 \geq 0\}.$$

Denote by ${}_1\mathcal{V}(X)$ the class of such series f_1. Furthermore, since $x_2 f_2(X)$ also does not contain negative powers of x_2, the coefficient f_{2Q} in $f_2(X)$ of (35) will vanish unless the vector Q lies either in ${}_1\mathbf{V}$, or along the ray $\{q_2 = -1,\ q_1 \geq r_{1*}\}$. Denote the class of series f_2 satisfying this property by ${}_2\mathcal{V}(X)$.

Sector ${}_1\mathbf{V}$ corresponds to the set

$${}_1\mathcal{U}(\varepsilon) = \{X : |X|^{R^*} \leq \varepsilon, |x_1| \leq \varepsilon\}, \tag{41}$$

and power series in ${}_1\mathcal{V}(X)$ are called convergent if they converge absolutely in some ${}_1\mathcal{U}(\varepsilon)$. Observe that ${}_1\mathbf{V}$ is contained in $\mathbf{V}$ and that ${}_1\mathcal{U}(\varepsilon)$ contains the sector $\mathcal{U}_{\mathbf{V}}(\varepsilon)$ given by (38).

Third normal form [6, Chapter II, Section 2, Theorem 2, p. 134]: If the series f_i in (35) are of class ${}_i\mathcal{V}(X)$, then there exists a formal change of coordinates (39), where the h_i are series of class ${}_i\mathcal{V}(Y)$, which transforms (35) into system (40) in which the g_i are series of class ${}_i\mathcal{V}(Y)$ consisting only of terms $g_{iQ}Y^Q$ satisfying $\langle Q, \Lambda\rangle = 0$.

Analogous statement also holds if we interchange the role of variables x_1 and x_2. Furthermore, it is shown in [6] that the behaviors of the integral curves of system (35) and the normal form (40) coincide in the region given by (41) similar to the Second Normal Form Theorem.

The advantage of the Third Normal Form over the Second Normal Form is that it describes the behavior of integral curves on a bigger region, albeit for a smaller class of power series.

Methods of integration of systems given in the aforementioned normal forms are carefully described in [6]. This makes it possible to construct the local phase portrait of these systems.

5.3 The Newton diagram

Let $\mathcal{X}$ be a real analytic vector field on $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{t} &= \sum_{j+k>1} \alpha_{jk} t^j s^k = t f_1(t,s), \\ \dot{s} &= \sum_{j+k>1} \beta_{jk} t^j s^k = s f_2(t,s). \end{aligned} \tag{42}$$

Of course, this notation for components of $\mathcal{X}$ is independent of the notation of Section 4, where f was the map defined in Section 4.2. We write

$$f_j(t,s) = \sum_{Q} f_{jQ}(t,s)^Q, \tag{43}$$

where $Q = (q_1, q_2)$ and $(t,s)^Q = t^{q_1}s^{q_2}$. *The support* $\mathbf{D}$ of $\mathcal{X}$ is the set of points $Q = (q_1, q_2)$ in $\mathbb{Z}^2$ such that $|f_{1Q}| + |f_{2Q}| \neq 0$. Fix a vector $P \in \mathbb{R}^2$ and put $c = \sup_{Q\in\mathbf{D}}\langle Q, P\rangle$; here $\langle \bullet, \bullet\rangle$

denotes the euclidean inner product. The set

$$L_P = \{Q \in \mathbb{R}^2 : \langle Q, P\rangle = c\}$$

forms the *support line* L_P of $\mathbf{D}$ with respect to the vector P, while the set

$$L_P^{(-)} = \{Q \in \mathbb{R}^2 : \langle Q, P\rangle \leq c\}$$

defines the *support half-space* $L_P^{(-)}$ corresponding to the vector P.

The Newton polygon Γ is defined as the intersection of all the support half-spaces of $\mathbf{D}$, that is,

$$\Gamma = \bigcap_{P\in\mathbb{R}^2\setminus\{0\}} L_P^{(-)}.$$

It coincides with the closure of the convex hull of $\mathbf{D}$ (see [6]). Its boundary consists of edges, which we denote by $\Gamma_j^{(1)}$, and vertices, which we denote by $\Gamma_j^{(0)}$, where j is some enumeration. In this notation the upper index expresses the dimension of the object.

Part of the boundary of Γ, called the *Newton diagram* or the *open Newton polygon* in the terminology of [6], denoted by $\hat{\Gamma}$, plays an important role in the theory of power series transformations. For simplicity we consider only the case relevant to us when $\mathbf{D}$ is contained in the set $\{Q = (q_1, q_2) : q_j \geq -1,\ \ j = 1, 2\}$. Then the Newton diagram can be constructed explicitly as follows. Let $q_{2*} = \min\{q_2 : (q_1, q_2) \in \mathbf{D}\}$. Then $x_2 = q_{2*}$ is the horizontal support line to $\mathbf{D}$. Set $q_{1*} = \min\{q_1 : (q_1, q_{2*}) \in \mathbf{D}\}$. The point $\Gamma_1^{(0)} := (q_{1*}, q_{2*})$ is the left boundary point of the intersection of $\mathbf{D}$ with the horizontal support line $q_2 = q_{2*}$. Consider the support line L_P for $\mathbf{D}$ through $\Gamma_1^{(0)}$ satisfying the following assumptions:

(i) $P = (p_1, p_2)$ with $p_1 < 0$ and $p_2 < 0$;
(ii) L_P contains at least one other point of $\mathbf{D}$.

The first assumption means that the line L_p admits a normal vector that lies in the third quadrant. In particular, L_P is not a horizontal or vertical line. Clearly, these two conditions define such a support line uniquely. If the line L_P does not exist, our procedure stops on this first step and we set $\hat{\Gamma} = \{\Gamma_1^{(0)}\}$, that is the Newton diagram consists of a single vertex. Otherwise, denote by $\Gamma_2^{(0)}$ the left boundary point of the intersection of $\mathbf{D}$ with L_P. Consider now the support line through $\Gamma_2^{(0)}$ with the aforementioned properties (i) and (ii); hence, it contains a point of $\mathbf{D}$ different from $\Gamma_1^{(0)}$. Continuing this procedure, we arrive at the point $Q^* = (q_1^*, q_2^*)$ which is the lowest point of $\mathbf{D}$ on the left vertical support line of $\mathbf{D}$, that is, $q_1^* = \min\{q_1 : (q_1, q_2) \in \mathbf{D}\}$ and $q_2^* = \min\{q_2 : (q_1^*, q_2) \in \mathbf{D}\}$. Denote this last point by $\Gamma_k^{(0)}$. For every $j = 1, \ldots, k-1$, we denote by $\Gamma_j^{(1)}$ the edge joining the vertices $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$. Thus by construction, the points $\Gamma_1^{(0)}$ and $\Gamma_k^{(0)}$ are joined by the Newton diagram $\hat{\Gamma}$.

It is important to notice here that all edges and vertices of the Newton diagram $\hat{\Gamma}$ are edges and vertices of the Newton polygon Γ, but in general, not all edges and vertices of Γ are edges and vertices of $\hat{\Gamma}$. Consider some examples.

Example 1. Let **D** consist of two points $(1, 1)$ and $(1, 2)$. Then the Newton diagram consists of a single vertex $\Gamma_1^{(0)} = (1, 1)$. □

The next example will occur in Section 6.

Example 2. Let **D** consist of three points $(2, 0)$, $(4, 0)$, and $(0, 2)$. Then the Newton diagram is formed by two vertices $\Gamma_1^{(0)} = (2, 0)$ and $\Gamma_2^{(0)} = (0, 2)$, and one edge $\Gamma_1^{(1)}$, which is the segment joining these vertices. □

5.4 Nonelementary singularity

Bruno's method for construction of the phase portrait of a vector field near a nonelementary singular point can be described as follows. For each element $\Gamma_j^{(d)}$ of the Newton diagram associated with (42), there is a corresponding sector $\mathcal{U}_j^d$ in the phase space $\mathbb{R}^2_{(t,s)}$, so that together they form a neighborhood of the origin (here boundaries of the sectors are not necessarily integral curves). In each $\mathcal{U}_j^0$ one brings the system to a normal form, and in $\mathcal{U}_j^1$ one uses power transformations (quasihomogeneous blow-ups) to reduce the problem to the study of elementary singularities of the transformed system. This allows one to determine the behavior of the orbits in each sector applying the Normal Form theorems discussed and using a careful study of integral curves for all types of normal forms in [6]. After that the results in each sector are glued together to obtain the overall phase portrait of the system near the origin.

We now consider some important special cases corresponding to particular elements of the Newton diagram.

Case of a vertex. Let $Q = \Gamma_j^{(0)}$ be a vertex of the Newton diagram. Consider the edges $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$ adjacent to Q in the Newton diagram. Next, consider the *unit* (i.e., their coordinates are coprime integers) vectors $R_{j-1} = (r_{1,j-1}, r_{2,j-1})$ and $R_j = (r_{1,j}, r_{2,j})$ directional to $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$, respectively. We impose here the restrictions $r_{2,j-1} > 0$ and

$r_{2,j} > 0$ so these vectors are determined uniquely. Set $R_* = -R_{j-1}$ and $R^* = R_j$. In the special case when Q is a boundary point of $\hat{\Gamma}$, one of the adjacent edges does not exist, so if Q is the right boundary point Q_*, we set $R_* = (1, 0)$, and if Q is the left boundary point Q^*, we put $R^* = (0, 1)$.

The method of [6] associates to Q a set defined by

$$\mathcal{U}_j^{(0)}(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{R^*} \leq \varepsilon, \ (|t|, |s|)^{R_*} \leq \varepsilon, |t| \leq \varepsilon, |s| \leq \varepsilon\}, \tag{44}$$

for some $\varepsilon > 0$. System (42), after the change of the old time variable τ with the new time variable τ_1 satisfying $d\tau_1 = (t, s)^Q \, d\tau$, is of form (35). Furthermore, the vectors R^* and R_* defined already by the adjacent edges at Q will generate for this new system (35) the convex cone $\mathbf{V}$ as described in the previous subsection, so the notation is consistent. The obtained system satisfies the assumptions of the Principal or the Second Normal Form Theorem. The behavior of the integral curves of the normal form and the original system coincides in $\mathcal{U}_j^{(0)}(\varepsilon)$ for ε sufficiently small.

A particularly simple case occurs when $Q = (q_1, q_2) = \Gamma_j^{(0)}$ is the first (i.e., the right) or the last (i.e., the left) point of $\hat{\Gamma}$, and Q is not contained in the first quadrant (Type I according to classification in [6, p. 138]). In this situation one of the coordinates of Q equals -1. Say, if $q_2 = -1$, that is, Q is the right point of $\hat{\Gamma}$, then one takes $R_* = (1, 0)$ according to the general rule stated already. The corresponding normal form has vertical integral curves. It follows that the original system (42) in the set

$$\mathcal{U}_*(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{R^*} \leq \varepsilon, |t| \leq \varepsilon\}$$

does not have any integral curves terminating at the origin. Similarly, if $q_1 = -1$, that is, if Q is the left point of Γ, then $R^* = (0, 1)$, and again in

$$\mathcal{U}^*(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{R_*} \leq \varepsilon, |s| \leq \varepsilon\}$$

the system does not have any characteristic orbits.

Case of an edge. Suppose now that $\Gamma_j^{(1)}$ is an edge of $\hat{\Gamma}$. Let $R = (r_1, r_2)$, $r_2 > 0$ be a unit directional vector of $\Gamma_j^{(1)}$. The corresponding set in the phase space is given by

$$\mathcal{U}_j^1(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : \varepsilon \leq (|t|, |s|)^R \leq 1/\varepsilon, \ |t| \leq \varepsilon, |s| \leq \varepsilon\}. \tag{45}$$

Consider the power transformation given by $y_1 = t^{k_1} s^{k_2}$, $y_2 = t^{r_1} s^{r_2}$, where the integers k_1 and k_2 are chosen such that the determinant of the matrix

$$A = \begin{pmatrix} k_1 & k_2 \\ r_1 & r_2 \end{pmatrix} \tag{46}$$

equals 1. In the matrix form, we can write $X = (t, s)$,

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

$$F_Q = \begin{pmatrix} f_{1q} \\ f_{2q} \end{pmatrix}.$$

Then (42) can be given by

$$(\ln \dot{X}) = \sum_{Q \in \mathbf{D}} F_Q X^Q, \tag{47}$$

where $X^Q = t^{q_1} s^{q_2}$. The power transformation can be expressed now as $Y = X^A$, taking (47) into

$$(\ln \dot{Y}) = \sum_{Q' \in D'} F'_{Q'} Y^{Q'},$$

with $Y = (y_1, y_2)$, $Q' = (A^t)^{-1} Q$, $D' = (A^t)^{-1} \mathbf{D}$ (the superscript t stands for transposition), and $F'_{Q'} = AF_Q$. After division by the maximal power of y_1, one obtains a new system. Here the y_2-axis corresponds to $\{t = s = 0\}$ in the original coordinates, and therefore, one needs to investigate the new system in a neighborhood of the y_2-axis. Quite often the topological behavior of the system in $\mathcal{U}_j^1(\varepsilon)$ can be determined by considering the *truncation* of the system which is obtained by taking the sum in (43) only over the vertices contained in $\Gamma_j^{(1)}$. The detailed discussion is in [6, pp. 140–141]. For instance, in the situation which we will encounter here, the truncated system will have an elementary singularity. In general, the singularities of the new system can be nonelementary, but they are simpler than those of the original system. Therefore, the general method described earlier can be applied and an induction procedure can be used.

We do not go into further details since the goal of this section is just to outline the strategy of the employed method. The computations of the next sections will strictly follow the presented method and, as we hope, will clarify the details.

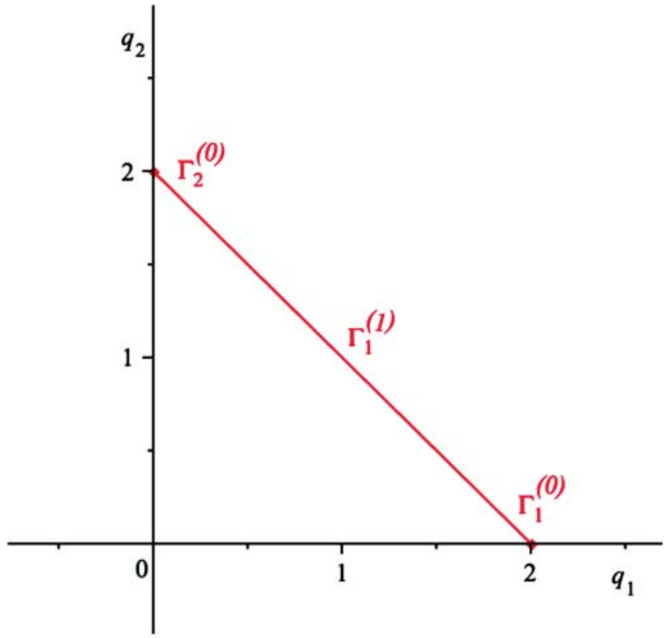

Fig. 1. The Newton diagram for (48).

6 Phase Portrait of the Standard Umbrella

Since the standard umbrella corresponds to the nongeneric case where ϕ is the identity map, we study its characteristic foliation separately. We rewrite system (15) in the form

$$\begin{aligned} \dot{t} &= t(-3t^2 - s^2 - 3t^4) = tf_1(t,s), \\ \dot{s} &= s(s^2 + 4t^2 + 7t^4) = sf_2(t,s), \end{aligned} \tag{48}$$

and set

$$f_j(t,s) = \sum_{Q} f_{jQ}(t,s)^Q,$$

where $Q = (q_1, q_2)$ is the multi-index with integer entries, and $(t,s)^Q = t^{q_1}s^{q_2}$.

The Newton diagram $\hat{\Gamma}$ consists of two vertices $\Gamma_1^{(0)} = (2,0)$ and $\Gamma_2^{(0)} = (0,2)$ and the line segment (edge) $\Gamma_1^{(1)}$ between them (see Figure 1). We point out that the point $(4,0)$ lies in the support $\mathbf{D}$ but does not belong to the Newton diagram $\hat{\Gamma}$. For each element of the Newton diagram (the two vertices and the edge), there is a corresponding sector in the phase space $\mathbb{R}^2_{(t,s)}$, so that together they form a neighborhood of the origin. Accordingly, we consider three cases.

Case 1. First consider the vertex $(2,0)$. Following the strategy outlined in Section 5.4, we set $R_* = (1,0)$ and $R^* = (-1,1)$. We can make the change of time $d\tau_1 = t^2 d\tau$. This yields the system

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}\tau_1} &= -t(3 + t^{-2}s^2 + 3t^2) = -3t + tf_1(t,s),\\ \frac{\mathrm{d}s}{\mathrm{d}\tau_1} &= s(4 + t^{-2}s^2 + 7t^2) = 4s + sf_2(t,s).\end{aligned} \tag{49}$$

The Newton diagram $\hat{\Gamma}$ corresponding to (49) has vertices $(-2,2)$ and $(2,0)$, and in particular, it is contained in the sector $\mathbf{V}$ (with the angle $< \pi$) bounded by the rays generated by R_* and R^*. Therefore, for sufficiently small ε, in the sector

$$\mathcal{U}_1^{(0)} = \{(t,s) \in \mathbb{R}^2 : (|t|,|s|)^{R_*} \leq \varepsilon,\ (|t|,|s|)^{R^*} \leq \varepsilon\} = \{|t| \leq \varepsilon,\ |s| \leq \varepsilon|t|\},$$

there exists a smooth change of variables (t,s) putting the initial system to the Second Normal Form of Bruno. In the new coordinates the system has the form

$$\begin{aligned}\dot{y}_1 &= -3y_1 + y_1 \sum g_{1Q}(y_1,y_2)^Q,\\ \dot{y}_2 &= 4y_2 + y_2 \sum g_{2Q}(y_1,y_2)^Q,\end{aligned} \tag{50}$$

where the coefficients g_{1Q} and g_{2Q} are all zero except those for which $-3q_1 + 4q_2 = 0$. The line $L := \{-3y_1 + 4y_2 = 0\}$ determined by the linear part of system (50) intersects the interior of the sector $\mathbf{V}$ (see Figure 2). It follows (see Bruno [6, p. 132]) that the system defined by (50), and hence by (49), is a saddle, that is, each ray $\{y_1 = 0, y_2 > 0\}$, $\{y_1 > 0, y_2 = 0\}$ is an integral curve, and in each quadrant in $\mathbb{R}^2$, the integral lines are homeomorphic to hyperbolas. This is the description of system (18) in sector $\mathcal{U}_1^{(0)}$.

Case 2. Consider now the second vertex $(0,2)$. Here we have $R_* = (1,-1)$ and $R^* = (0,1)$ (Figure 3). The corresponding sector where the change of dependent variables will be performed is given by

$$\mathcal{U}_2^{(0)} = \{(t,s) \in \mathbb{R}^2 : (|t|,|s|)^{R_*} \leq \varepsilon,\ (|t|,|s|)^{R^*} \leq \varepsilon\} = \left\{|s| \leq \varepsilon,\ |s| \geq \frac{|t|}{\varepsilon}\right\}.$$

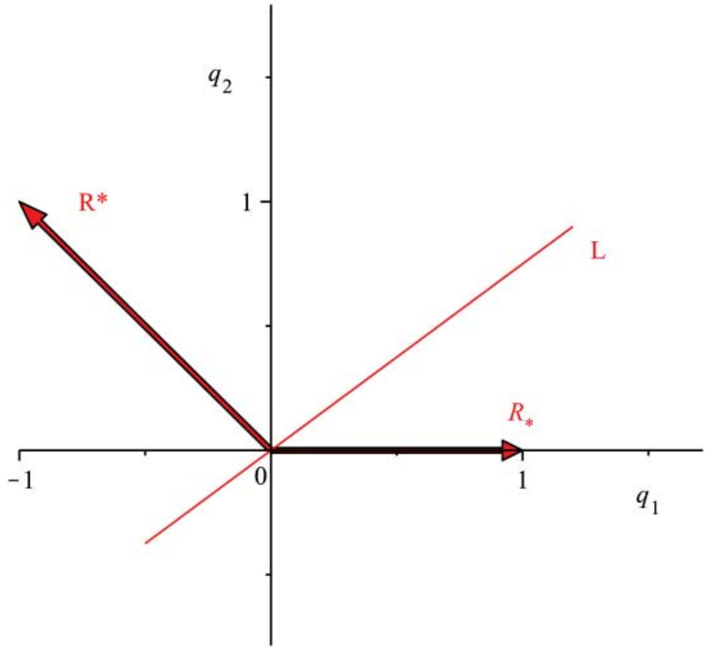

Fig. 2. Case 1 for (48).

The change of time $d\tau_1 = s^2\, d\tau$ transforms system (18) into

$$\frac{dt}{d\tau_1} = -t + t(3t^2 s^{-2} + 3t^4 s^{-2}),$$
$$\frac{ds}{d\tau_1} = s + s(4t^2 s^{-2} + 7t^4 s^{-2}). \tag{51}$$

As just seen, there exists a smooth change of variables (t, s) putting this system to the second normal form:

$$\dot{y}_1 = -y_1 + y_1 \sum g_{1Q}(y_1, y_2)^Q,$$
$$\dot{y}_2 = y_2 + y_2 \sum g_{2Q}(y_1, y_2)^Q,$$

where the coefficients g_{1Q} and g_{2Q} are all zero except those that belong to the line $L := \{-q_1 + q_2 = 0\}$. This line intersects the sector **V** bounded by R_* and R^* which implies that this system is again a saddle. This gives the phase portrait of (18) in sector $\mathcal{U}_2^{(0)}$

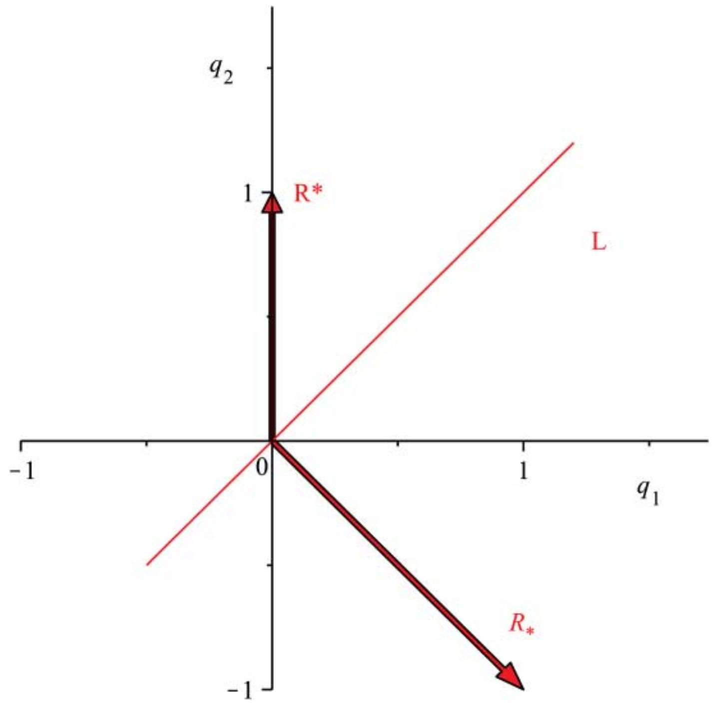

Fig. 3. Case 2 for (48).

Case 3. The remaining case of the edge between (2, 0) and (0, 2) will correspond to the sector $\mathcal{U}_1^{(1)}$, which is the complement of $\mathcal{U}_1^{(0)} \cup \mathcal{U}_2^{(0)}$. We make the following change of variables

$$y_1 = t,$$
$$y_2 = t^{-1}s. \tag{52}$$

In the matrix form, we write $X = (t, s)$, and the change of variables (52) can be expressed as $Y = X^A$ with the matrix of exponents

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then system (18) takes the form

$$\dot{y}_1 = y_1(-3y_1^2 - y_1^2 y_2^2 - 3y_1^4),$$
$$\dot{y}_2 = y_2(7y_1^2 + 2y_1^2 y_2^2 + 10y_1^4).$$

The edge of $\hat{\Gamma}$ becomes vertical in the new system. Performing as before a change of time, we may divide both sides by y_1^2 to obtain

$$\begin{aligned} \dot{y}_1 &= -3y_1 - y_1 y_2^2 - 3y_1^3 = y_1(-3 - y_2^2 - 3y_1^2), \\ \dot{y}_2 &= 7y_2 + 2y_2^3 + 10y_1^2 y_2 = y_2(7 + 2y_2^2 + 10y_1^2). \end{aligned} \tag{53}$$

Under the change of variables (52), the line $y_1 = 0$ corresponds to the origin, and therefore, we are interested in the integral curves of system (53) that intersect the line $y_1 = 0$ at points with $y_2 \neq 0$. The set $\{y_1 = 0, \ \pm y_2 > 0\}$ are integral curves of (53), but they correspond to $t = s = 0$ in the original system. According to Bruno [6, p. 141], the points on the y_2 axis can be either simple points, in which case the integral curves of (53) near such points are parallel to the y_2-axis, or singular points. The truncation of system (53) (see the end of the previous section) contains only the terms that correspond to the edge under consideration and its vertices, and thus has the form

$$\begin{aligned} \dot{y}_1 &= y_1 \hat{f}'_{\hat{1}0}(y_2), \\ \dot{y}_2 &= y_2 \hat{f}'_{\hat{2}0}(y_2), \end{aligned} \tag{54}$$

where $\hat{f}'_{\hat{2}0}(y_2) = 7 + 2y_2^2$ (we follow the notation of [6]). Singular points are determined from the equation $\hat{f}'_{\hat{2}0}(y_2) = 0$. In our case $\hat{f}'_{\hat{2}0}(y_2)$ is strictly positive. Therefore, in (54) all points with $y_1 = 0$, $y_2 \neq 0$ are simple points. From this we conclude that in the sector $\mathcal{U}_1^{(1)}$ no integral curves of system (18) intersect the origin.

With this information the integral curves in all sectors can be glued together. It is readily verified that the phase portrait of system (18) is in fact a saddle, the integral curves in each quadrant of $\mathbb{R}^2$ are homeomorphic to hyperbolas and do not intersect the coordinate axes (see Figure 4).

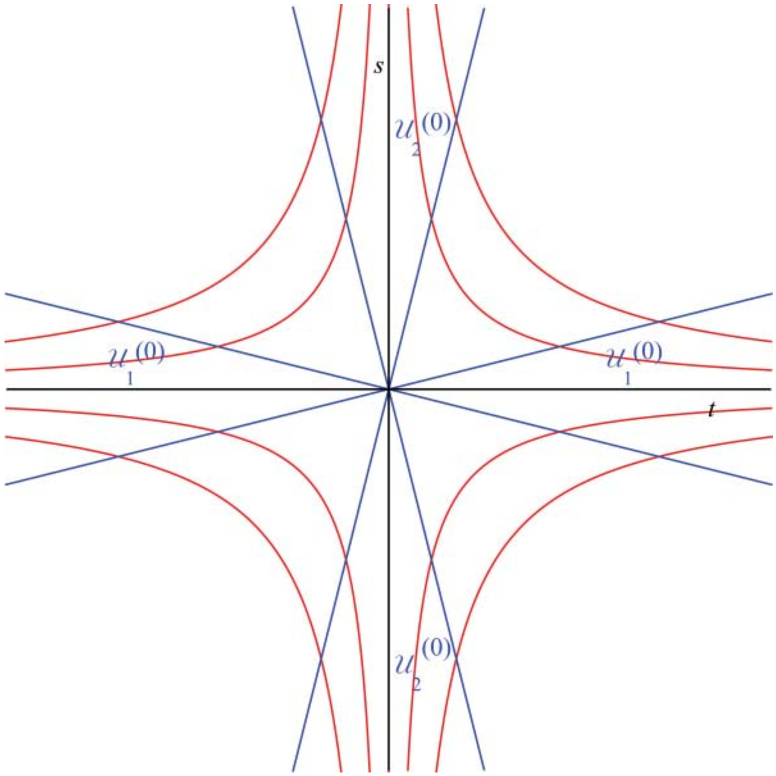

Fig. 4. Phase portrait of (48).

7 Phase Portrait of Umbrella in General Position

We now perform similar calculations for the algorithm to determine the topological structure near the origin of the dynamical system defined by (29). First of all we represent it in the canonical form

$$\begin{aligned} \dot{t} &= t(-2g_{12}s + \alpha_{02}t^{-1}s^2 - 3g_{22}t^2 + o(|s| + |t^{-1}s^2| + |t|^2)), \\ \dot{s} &= s(4g_{11}t^2 + \beta_{12}ts + \beta_{03}s^2 + 6g_{12}t^4s^{-1} + o(|t^2| + |ts| + |s^2| + |t^4s^{-1}|)). \end{aligned} \tag{55}$$

The Newton diagram $\hat{\Gamma}$ consists of three vertices $(-1, 2)$, $(0, 1)$, and $(4, -1)$, and the two edges between them (Figure 5). Five cases should be considered each corresponding to a vertex or an edge of $\hat{\Gamma}$.

Case 1. Vertex $(4, -1)$. This corresponds to the situation discussed in Section 5.4. We obtain immediately the behavior of integral curves of the system. Namely, in the

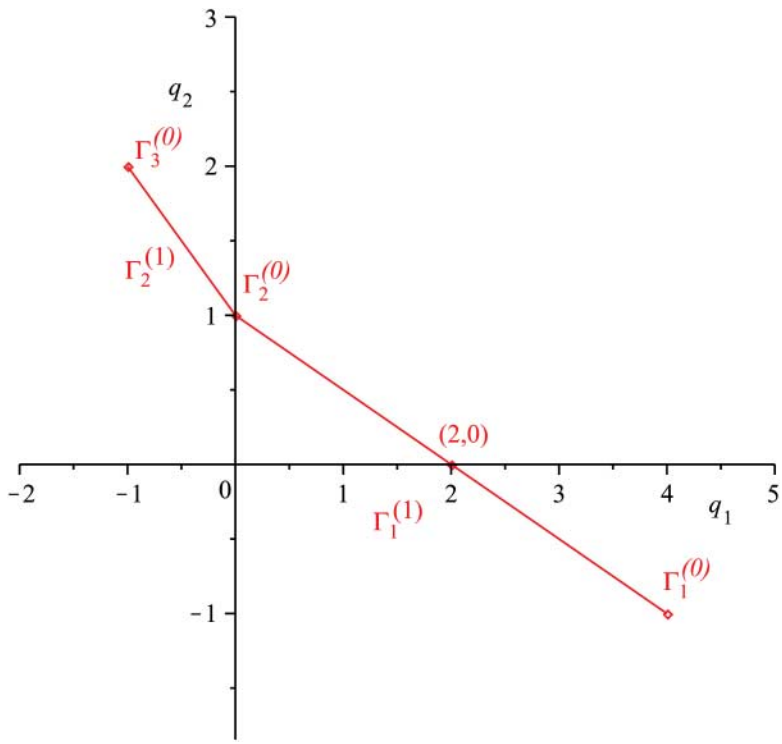

Fig. 5. The Newton diagram for (55).

sector

$$\mathcal{U}_1^{(0)} = \{(t,s) \in \mathbb{R}^2 : (|t|,|s|)^{(1,0)} \le \varepsilon,\ (|t|,|s|)^{(-2,1)} \le \varepsilon\} = \{|t| \le \varepsilon,\ |s| \le \varepsilon |t|^2\}$$

the integral curves are vertical, in particular, there are no curves terminating at the origin.

Case 2. Vertex $(-1,2)$. Again the same analysis works here. Since $(-1,2)$ is the end point of $\hat{\Gamma}$, that is, of Type I in [6, p. 138], it follows from [6] that in

$$\mathcal{U}_3^{(0)} = \{(t,s) \in \mathbb{R}^2 : (|t|,|s|)^{(0,1)} \le \varepsilon,\ (|t|,|s|)^{(1,-2)} \le \varepsilon\} = \{|s| \le \varepsilon,\ |t| \le \varepsilon |s|^2\}$$

the integral curves are horizontal, and no curves terminate at the origin.

Case 3. Vertex (0, 1). This is Type III in [6, p. 139]. After a change of time so that $d\tau_1 = s\, d\tau$, the system takes the form

$$\begin{aligned} \dot{t} &= t(-2g_{12} + \alpha_{02} t^{-1} s - 3g_{22} t^2 s^{-1} + o(1 + |t^{-1}s| + |t^2 s^{-1}|)), \\ \dot{s} &= s(4g_{11} t^2 s^{-1} + \beta_{12} t + \beta_{03} s + 6g_{12} t^4 s^{-2} + o(|t^2 s^{-1}| + |t| + |s| + |t^4 s^{-2}|)). \end{aligned} \tag{56}$$

There are two sectors which can be assigned to vertex (0, 1). One of them is determined by $R_* = (2, -1)$ and $R^* = (-1, 1)$, and equals

$$\mathcal{U}_2^{(0)} = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{R^*} \le \varepsilon,\ \ (|t|, |s|)^{R_*} \le \varepsilon\}.$$

We may apply here the Second Normal Form of Bruno. Since we consider a generic case, we have $\lambda_1 = -2g_{12} \neq 0$. Further, $\lambda_2 = 0$, because the second equation has no free term. Recall that we use the notation $\Lambda = (\lambda_1, \lambda_2)$. The line L determined by

$$L = \{Q = (q_1, q_2) \in \mathbb{R}^2 : \langle Q, \Lambda \rangle = 0\} = \{q_1 = 0\} \tag{57}$$

enters the interior of the sector bounded by R^* and R_*. It follows that in $\mathcal{U}_2^{(0)}$ there are no integral curves terminating at the origin.

On the other hand, we may use the Third Normal Form of Bruno for (56). It is valid on a bigger domain, namely, on

$${}_2\mathcal{U}_2^{(0)} = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{R_*} \le \varepsilon,\ |s| \le \varepsilon\} = \{|t|^2 \le \varepsilon |s|,\ |s| \le \varepsilon\}.$$

The region of the (t, s)-space where the dynamics takes place is given by

$${}_2\mathbf{V} = \{Q : Q = a_1 R_* + a_2 \cdot (0, 1),\ \ a_1, a_2 \ge 0\}.$$

Now the line L determined from (57) enters ${}_2\mathbf{V}$ along its boundary, the s-axis. In general, this yields a complicated behavior of the system in ${}_2\mathcal{U}_2^{(0)}$. In fact, there are four possibilities as described in [6, p. 134 Case c)]. So which case is it? The salvation comes from Case 2 seen earlier: it describes the behavior of the system in $\mathcal{U}_3^{(0)}$ (which is a subset of ${}_2\mathcal{U}_2^{(0)}$ and a neighborhood of the s-axis). According to Case 2, the integral curves are horizontal near the s-axis, which eliminates all possibilities but one. We conclude that no integral curves enter the origin in ${}_2\mathcal{U}_2^{(0)}$.

Case 4. Edge connecting (0, 1) and (−1, 2). The corresponding sector is defined by

$$\mathcal{U}_2^{(1)} = \{Q \in \mathbb{R}^2 : \varepsilon \le (|t|, |s|)^{(-1,1)} \le \varepsilon^{-1}\}$$

(see [6, p. 139]). This case is subsumed by Case 3 because $\mathcal{U}_2^{(1)} \subset {}_2\mathcal{U}_2^{(0)}$ in a suitable neighborhood of the origin.

Case 5. Edge connecting (0, 1) and (4, −1). We will consider the truncation of system (29), that is, we keep only terms that are related to the edge under consideration. We have

$$\begin{aligned} \dot{t} &= t(-2g_{12}s - 3g_{22}t^2), \\ \dot{s} &= s(-4g_{11}t^2 + 6g_{12}t^4 s^{-1}). \end{aligned} \tag{58}$$

The directional vector is $R = (-2, 1)$, and the sector in which the dynamics should be understood is

$$\mathcal{U}_1^{(1)} = \left\{(t, s) : \varepsilon \le (|t|, |s|)^{(-2,1)} \le \frac{1}{\varepsilon}, |t|, |s| \le \varepsilon\right\} = \left\{\varepsilon|t|^2 \le |s|, |s| \le \frac{1}{\varepsilon}|t|^2, |t| \le \varepsilon, |s| \le \varepsilon\right\}. \tag{59}$$

We need to make the following change of coordinates:

$$\begin{aligned} y_1 &= t, \\ y_2 &= t^{-2}s, \end{aligned} \tag{60}$$

which corresponds to the matrix

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

In the new coordinate system, (58) becomes

$$\begin{aligned} \dot{y}_1 &= y_1(-2g_{12}y_1^2 y_2 - 3g_{22}y_1^2), \\ \dot{y}_2 &= y_2(4g_{12}y_1^2 y_2 + (6g_{22} + 4g_{11})y_1^2 + 6g_{12}y_1^2 y_2^{-1}). \end{aligned} \tag{61}$$

We divide by the maximal power of y_1, which equals 2 in this case, by performing the change of the independent variable: $d\tau_1 = y_1^2\, d\tau$. This yields

$$\begin{aligned}\dot{y}_1 &= y_1(-2g_{12}y_2 - 3g_{22}),\\ \dot{y}_2 &= y_2(4g_{12}y_2 + (6g_{22} + 4g_{11}) + 6g_{12}y_2^{-1}).\end{aligned} \tag{62}$$

This is the system of Type I in [6, p. 125]. The y_2-axis is an integral curve, but it corresponds to the origin in (58). Consider first the points where the expression $4g_{12}y_2^2 + (6g_{22} + 4g_{11})y_2 + 6g_{12}$ is not zero; the integral curves near such a point are parallel to the y_2-axis. Going back to the original system via the inverse transformation to (60), we see that the y_2-axis blows down to the origin. Hence, these integral curves do not terminate at zero in the original system. Now we need to investigate the situation near points where the aforementioned expression vanishes. For this we solve the quadratic equation

$$2g_{12}y_2^2 + (3g_{22} + 2g_{11})y_2 + 3g_{12} = 0. \tag{63}$$

The discriminant of this equation is

$$\mathcal{D} = 4g_{11}^2 + 9g_{22}^2 + 12g_{11}g_{22} - 24g_{12}^2.$$

Since $4g_{11}^2 + 9g_{22}^2 \geq 12g_{11}g_{22}$, it follows that $\mathcal{D} \geq 24g_{11}g_{22} - 24g_{12}^2 = 24\Delta > 0$. Here Δ is defined by (8). Thus, Equation (63) always has two simple roots:

$$c_{\pm} = \frac{-(3g_{22} + 2g_{11}) \pm \sqrt{4g_{11}^2 + 9g_{22}^2 + 12g_{11}g_{22} - 24g_{12}^2}}{4g_{12}}.$$

(since we consider the generic case, we can assume that $g_{12} \neq 0$). We point out that $c_{\pm}$ are either both positive or both negative.

We need to investigate the dynamics near each point $(0, c_{\pm})$. For that we first need to translate $c_{\pm}$ to the origin via

$$z_1 = y_1, \quad y_2 = c_{\pm} + z_2.$$

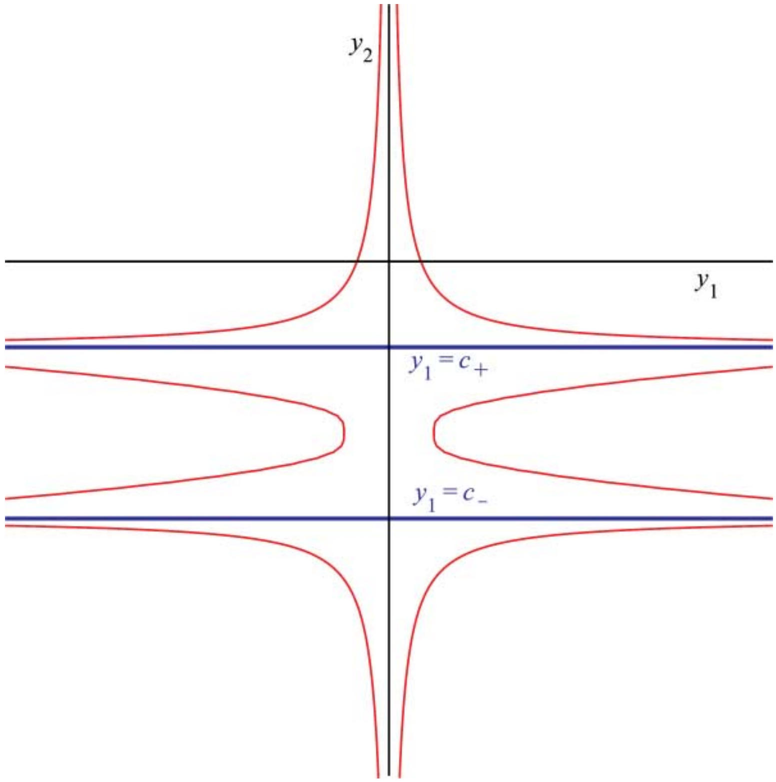

Fig. 6. Phase portrait in y-coordinates for $g_{12} > 0$.

In the new coordinates, the system becomes

$$\begin{aligned} \dot{z}_1 &= z_1(-(2g_{12}c_\pm + 3g_{22}) - 2g_{12}z_2), \\ \dot{z}_2 &= z_2((8g_{12}c_\pm + 6g_{22} + 4g_{11}) + 4g_{12}z_2). \end{aligned} \tag{64}$$

This is a system for which the origin is an elementary singularity (the linear part is not zero). To determine the dynamics, we need to understand the sign of the coefficients of the linear part, that is, of

$$\lambda_1 = -(2g_{12}c_\pm + 3g_{22}) = -\tfrac{3}{2}g_{22} + g_{11} \mp \tfrac{1}{2}\sqrt{4g_{11}^2 + 9g_{22}^2 + 12g_{11}g_{22} - 24g_{12}^2}$$

and

$$\lambda_2 = 8g_{12}c_\pm + 6g_{22} + 4g_{11} = \pm 2\sqrt{4g_{11}^2 + 9g_{22}^2 + 12g_{11}g_{22} - 24g_{12}^2}.$$

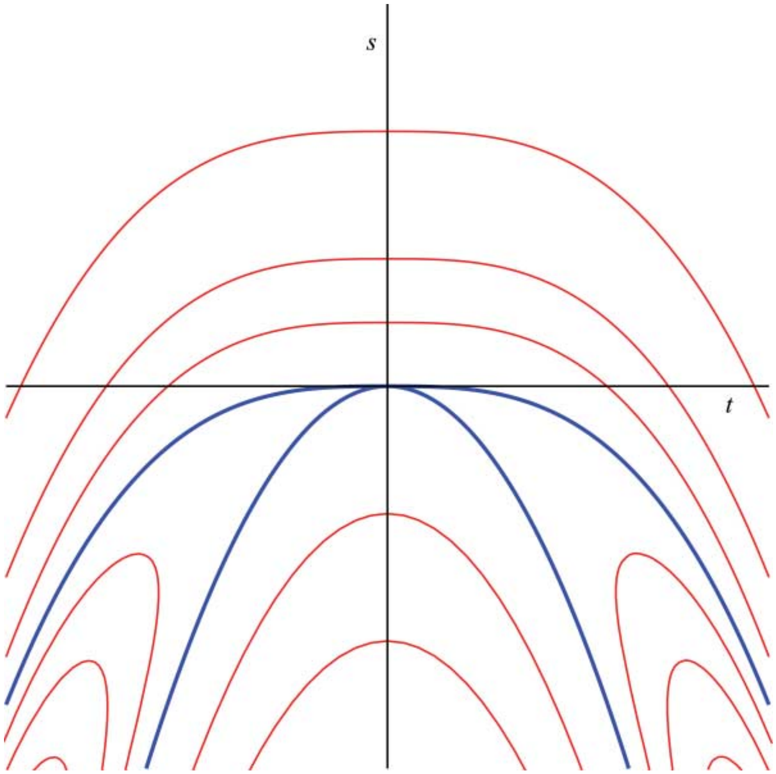

Fig. 7. Phase portrait of (55), $g_{12} > 0$.

Claim. λ_1 and λ_2 are of opposite signs both for c_+ and c_-. □

First note that λ_1 and λ_2 depend only on the coefficients g_{jk}, that is, only on the linear part of the map $\psi \circ \phi$. Therefore, it is enough to prove the claim for linear symplectomorphisms. If ϕ is the identity map, then it is easy to see that λ_1 and λ_2 are of opposite signs.

Suppose that for some linear symplectic map ϕ_0, the signs of λ_1 and λ_2 are the same. Since the symplectic group is connected, there is a path $\gamma \subset \mathrm{Sp}(4, \mathbb{R})$ connecting the identity and ϕ_0, and since λ_j depend continuously on ϕ, there exists a symplectic map on γ for which one of the λ_j is zero. Since $\mathcal{D} > 0$, it has to be λ_1. So $-\frac{3}{2}g_{22} + g_{11} = \pm\frac{1}{2}\sqrt{\mathcal{D}}$. Therefore,

$$4g_{11}^2 - 12g_{11}g_{22} + 9g_{22}^2 = 4g_{11}^2 + 9g_{22}^2 + 12g_{11}g_{22} - 24g_{12}^2.$$

This implies that $\Delta = 0$ – contradiction. This proves the claim.

Since λ_j are of different signs, it follows that both for c_+ and c_-, system (64) is a saddle at the origin. Now we are able to describe the overall dynamics in $\mathcal{U}_1^{(1)}$. In (y_1, y_2)-coordinates, we have the following: y_2-axis as well as the lines $y_2 = c_+$ and $y_2 = c_-$ are the integral curves. More precisely, the integral curves are six half-lines: $L_1 = \{(y_1, c_+), y_1 > 0\}$, $L_2 = \{(y_1, c_+), y_1 < 0\}$, $L_3 = \{(y_1, c_-), y_1 > 0\}$, $L_4 = \{(y_1, c_-), y_1 < 0\}$, $L_5 = \{(0, y_2) : y_2 > c_+\}$, and $L_6 = \{(0, y_2) : y_2 < c_-\}$, and one interval $I = \{(0, y_2) : \min\{c_-, c_+\} < y_2 < \max\{c_-, c_+\}\}$. The phase portraits near the points $(0, c_+)$ and $(0, c_-)$ are saddles, whose orbits in between the lines $y_2 = c_+$ and $y_2 = c_-$ are glued together, and are asymptotic to L_1 and L_3 or to L_2 and L_4; they do not touch I. Other orbits are asymptotic to L_2 and L_5 or to L_5 and L_1 or to L_6, L_4 or, finally, to L_6 and L_3 (see Figure 6). Going back to the original system via the inverse transformation to (60), we see that the y_2-axis blows down to a point, and we have two integral curves $s = c_\pm t^2$ entering the origin, while other integral curves are contained in the compliment of these two curves. Now, if we choose $\varepsilon > 0$ sufficiently small in (59), we see that both curves $s = c_\pm t^2$ enter $\mathcal{U}_1^{(1)}$. This completes Case 5.

Now if we combine all the five cases together, and glue the integral curves from all the cases, we see that the phase portrait at the origin of system (29) is a saddle (Figure 7). With this analysis we can now conclude the proof of Proposition 1. Indeed, let γ_1 and γ_2 be the curves $s = c_\pm t^2$. If K is a small compact not contained in the union of γ_1 and γ_2, then one of the hyperbolas of the characteristic foliation will touch K at some point. This proves Proposition 1.

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