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Filtrations, Hyperbolicity, and Dimension for Polynomial Automorphisms of $\mathbb{C}^n$

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1. Introduction

Let f be a polynomial automorphism of $\mathbb{C}^n$. We denote by $\hat{f}$ the natural extension of f to a meromorphic map in $\mathbb{P}^n$. Let I^+ denote the indeterminacy set of $\hat{f}$. Analogously, we denote by I^- the indeterminacy set of $\widehat{f^{-1}}$. We say that f is *regular* if f has degree greater than 1 and if $I^+ \cap I^- = \emptyset$.

In the case $n = 2$, the class of regular automorphisms consists of polynomial automorphisms with nontrivial dynamics—that is, finite compositions of generalized Hénon maps (see e.g. [BS1; FM; FS]). In fact, regular polynomial automorphisms can be considered as a natural generalization of complex Hénon maps to higher dimensions. Higher-dimensional regular maps are for instance the so-called shift-like automorphisms studied by Bedford and Pambuccian [BP]. For further examples we refer to Section 2. We point out that, unlike the two-dimensional case, for $n > 2$ there exist polynomial automorphisms with nontrivial dynamics that are not regular (see e.g. [CF]).

The notion of regular polynomial automorphisms was introduced by Sibony [Si], who comprehensively studied these maps using, in particular, methods from pluripotential theory.

In this paper we study the dynamics of regular polynomial automorphisms from a different point of view: We introduce the notion of hyperbolicity for a regular polynomial automorphism f and study its dynamics. That is, we classify the orbits of f analogously to the case of complex Hénon maps in [BS1]. Finally, we study the Hausdorff and box dimension of the Julia sets of f. We derive estimates for these dimensions in the hyperbolic as well as in the nonhyperbolic case.

We will now describe our results in more detail.

Let f be a regular polynomial automorphism of $\mathbb{C}^n$. We define $K^{\pm} = \{p \in \mathbb{C}^n : \{f^{\pm k}(p) : k \in \mathbb{N}\}$ is bounded$\}$ and the filled-in Julia set by $K = K^+ \cap K^-$. Furthermore, we define the sets $J^{\pm} = \partial K^{\pm}$ and $J = J^+ \cap J^-$. The set $J^{\pm}$ is called the forward/backward Julia set, and J is the Julia set of f (see Section 2 for details).

We construct a filtration of $\mathbb{C}^n$ that has particular escape properties for the orbits of f (see Proposition 3.1). For $n = 2$, the existence of a filtration was already

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shown by Bedford and Smillie [BS1]. In this case it proved to be a useful tool for analyzing the dynamics of f.

We apply our filtration to study hyperbolic maps. We say that f is *hyperbolic* if its Julia set J is a hyperbolic set and the periodic points are dense in J. It turns out that hyperbolicity implies that f is Axiom A (see Corollary 4.5). The latter is the classical notion for hyperbolic diffeomorphisms. We obtain a complete description for the possibilities of the orbits in the case of a hyperbolic map f. The following result is a consequence of Proposition 4.1 and Theorems 4.2 and 4.4.

THEOREM 1.1. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$, and let p be a point in $\mathbb{C}^n$. Then one of the following exclusive properties holds.*

(i) *There exists $q \in J$ such that $|f^k(p) - f^k(q)| \to 0$ as $k \to \infty$.*
(ii) *There exists an attracting periodic point α of f such that $|f^k(p) - f^k(\alpha)| \to 0$ as $k \to \infty$.*
(iii) *$\{f^k(p) : k \in \mathbb{N}\}$ converges to ∞ as $k \to \infty$.*

The inverse of f is also a regular polynomial automorphism; therefore, Theorem 1.1 also holds for f^{-1}. However, since f has constant Jacobian, attracting periodic points can only exist either for f or for f^{-1}. Theorem 1.1 implies that, in order to understand the "complicated" dynamics of a hyperbolic map, it is sufficient to understand the dynamics on its Julia set.

In the second part of this paper we derive estimates for the Hausdorff and box dimension of the Julia sets. Let d be the degree of f; then we denote by $l-1$ the dimension of I^- (as an algebraic variety). We show that J carries the full entropy of f, that is, $h_{\text{top}}(f|J) = l \log d$ (see Theorem 5.1), where h_{top} denotes the topological entropy. As a consequence, the upper box dimension of the Julia set is strictly positive (see Corollary 5.6). On the other hand, if f is not volume-preserving then the upper box dimension of K is strictly smaller than $2n$ (see Corollary 5.5).

It is a widely studied problem in one-dimensional complex dynamics to determine whether the Hausdorff dimension of the Julia set of a rational map is strictly less than 2 (see [Ur] for an overview). We solve the analogous problem in the case of hyperbolic regular polynomial automorphisms of $\mathbb{C}^n$, $n \geq 2$; namely, we show that the Hausdorff dimension of $J^{\pm}$ is strictly less than $2n$. More precisely, we derive an upper bound for the Hausdorff dimension of $J^{\pm}$ that is given in terms of topological pressure (see Theorem 5.13). This upper bound is strictly smaller than $2n$. Our theorem improves a result of Bowen [Bo], who showed that $J^{\pm}$ has zero Lebesgue measure. It should be noted that, for $n = 2$, it is possible to construct hyperbolic maps whose forward/backward Julia sets have Hausdorff dimension arbitrarily close to $4 = 2n$ (see [Wo3]).

This paper is organized as follows. In Section 2 we present basic facts about regular polynomial automorphisms of $\mathbb{C}^n$. In Section 3 we construct a filtration with particular escape properties for the orbits. Section 4 is devoted to the analysis of hyperbolic maps. Finally, we study in Section 5 the Hausdorff and the box dimension of the Julia sets.

For $n = 2$ it is shown in [BS1] that hyperbolicity of J already implies that f is a hyperbolic map; that is, J being hyperbolic implies the density of the periodic points in J. In particular, this provides a weaker definition of a hyperbolic map in the case $n = 2$. It would be interesting to know whether the analogous result holds for $n > 2$. See also the remark at the end of Section 4.

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2. Regular Polynomial Automorphisms

In this section we give an introduction to the dynamics of regular polynomial automorphisms of $\mathbb{C}^n$. This class of maps was studied in detail by Sibony [Si]. We refer to this article for proofs of the results presented here.

Let f be a polynomial automorphism of $\mathbb{C}^n$, $n \geq 2$. Then f admits an extension to a meromorphic map $\hat{f}\colon \mathbb{P}^n \to \mathbb{P}^n$. Let $\pi\colon \mathbb{C}^{n+1} \to \mathbb{P}^n$ be the canonical projection and let F be the homogeneous polynomial map in $\mathbb{C}^{n+1}$ corresponding to $\hat{f}$ (i.e., $\hat{f} = \pi \circ F \circ \pi^{-1}$). Then $I^+ = \pi \circ F^{-1}(0)$ is the indeterminacy set of $\hat{f}$. Analogously, I^- denotes the indeterminacy set of $\widehat{f^{-1}}$. The sets I^+ and I^- are algebraic varieties in $\mathbb{P}^n$ of codimension at least 2 contained in the hypersurface at infinity, which is denoted by H_0.

We write $f = (f_1, \dots, f_n)$. Let $\deg f_i$ denote the polynomial degree of f_i. Then $d = \deg f = \max\{\deg f_1, \dots, \deg f_n\}$ is the degree of f. A polynomial automorphism f of $\mathbb{C}^n$ is called *regular* if $d > 1$ and $I^+ \cap I^- = \emptyset$.

Throughout this paper, f will be a regular polynomial automorphism of $\mathbb{C}^n$. Note that the complex Jacobian $\det Df$ is constant in $\mathbb{C}^n$. Therefore, we can restrict our considerations to the volume-decreasing case ($|\det Df| < 1$) and to the volume-preserving case ($|\det Df| = 1$), as otherwise we simply consider f^{-1}.

Examples of regular polynomial automorphisms include the well-known family of generalized Hénon maps in $\mathbb{C}^2$. Moreover, the shift-like polynomial automorphisms in $\mathbb{C}^n$ studied by Bedford and Pambuccian [BP] have the property that a certain iterate is regular. Finally, certain classes of quadratic polynomial automorphisms of $\mathbb{C}^3$ studied by Fornæss and Wu [FW] contain concrete examples of regular maps.

For a regular polynomial automorphism f of $\mathbb{C}^n$, there exists an integer $l > 0$ such that $(\deg f)^l = (\deg f^{-1})^{n-l}$. Moreover, it follows that for every regular polynomial automorphism we have $\dim I^- = l - 1$ and $\dim I^+ = n - l - 1$.

Let $K^\pm$, K, $J^\pm$, and J be defined as in Section 1. Then all of these are f-invariant sets, and

$$\overline{K^\pm} = K^\pm \cup I^\pm, \tag{1}$$

where $\overline{K^{\pm}}$ denotes the closure in $\mathbb{P}^n$. Although $K^{\pm}$ and $J^{\pm}$ are closed and unbounded in $\mathbb{C}^n$, the sets K and J are compact. If $n = 2$, then the set $J^{\pm}$ is connected (see [BS2]).

Furthermore, the distance between $f^k(p)$ and I^- tends uniformly to zero on compact subsets of $\mathbb{C}^n \setminus K^+$. On the other hand, the distance between $f^k(p)$ and K tends uniformly to zero on compact subsets of K^+, and the family $\{f^k : k \in \mathbb{N}\}$ is equicontinuous in $\operatorname{int} K^+$, the interior of K^+. The analogous properties hold for f^{-1} and K^-.

3. Filtration Properties

In this section we construct for a regular polynomial automorphism f a filtration in $\mathbb{C}^n$ that exhibits particular escape properties for the orbits of f. Our approach is motivated by the work of Bedford and Smillie [BS1, Sec. 2] in the case of generalized Hénon maps.

PROPOSITION 3.1 (Filtration). *Let f be a regular polynomial automorphism of $\mathbb{C}^n$. Then there exist a compact set $V \subset \mathbb{C}^n$ with $K \subset \operatorname{int} V$ and sets $V^+, V^- \subset \mathbb{C}^n$ such that $\mathbb{C}^n = V \cup V^- \cup V^+$ is a disjoint union and the following inclusions hold*:

(i) $f(V^-) \subset V^-$;
(ii) $f(V^- \cup V) \subset V^- \cup V$;
(iii) $f^{-1}(V^+) \subset V^+$;
(iv) $f^{-1}(V^+ \cup V) \subset V^+ \cup V$.

Proof. Let V be a closed polydisk of sufficiently large radius such that $K \subset \operatorname{int} V$. Let $\hat{V}^+$ and $\hat{V}^-$ be open sets in $\mathbb{P}^n$ for which the following properties hold:

$$\begin{aligned} &\text{(a)}\ \ I^+ \subset \hat{V}^+,\ \ I^- \subset \hat{V}^-, \\ &\text{(b)}\ \ \widehat{f^{-1}}(\hat{V}^+) \subset \hat{V}^+,\ \ \hat{f}(\hat{V}^-) \subset \hat{V}^-, \\ &\text{(c)}\ \ \hat{V}^{\pm} \cap V = \emptyset,\ \ \hat{V}^{\pm} \cap K^{\mp} = \emptyset. \end{aligned} \tag{2}$$

We note that property (b) can be satisfied, since I^+ is an attracting set for $\widehat{f^{-1}}$ and I^- is an attracting set for $\hat{f}$ (see [Si, Prop. 2.2.6]). Also, $\hat{V}^{\pm}$ can be chosen to satisfy the second identity in (c) because of (1).

Let $V_0^{\pm} = \hat{V}^{\pm} \cap \mathbb{C}^n$. To construct V^+ and V^- we define

$$V_k^+ = f^k(V_0^+), \qquad V_k^- = f^{-k}(V_0^-), \tag{3}$$

where $k \geq 1$ and, at each step, if $V_k^+ \cap V_k^- \neq \emptyset$ then we replace the set V_0^- with $f^k(V_k^- \setminus V_k^+)$. To emphasize that V_0^- depends on the number of times this process is applied, we use the notation $V_0^-(k)$, where k is the number of iterations.

We also note that the shrinking of $V_0^-(k)$ does not change the union of V_k^+ and V_k^-, and that V_k^+ and V_k^- are disjoint for every $k > 0$. To prove the proposition we will show that there exists an $N > 0$ such that $\mathbb{C}^n \setminus V \subset V_N^+ \cup V_N^-$. We will do this in two steps.

Claim 1. There exists $N > 0$ *such that, for every* $p \in \mathbb{C}^n \setminus V$ *and every* $k \geq N$, *either* $f^k(p) \in \hat{V}^- \cap \mathbb{C}^n$ *or* $f^{-k}(p) \in \hat{V}^+ \cap \mathbb{C}^n$.

In order to prove the claim, we consider a neighborhood U in $\mathbb{P}^n$ of the compact set $H_0 \setminus (\hat{V}^- \cup \hat{V}^+)$. We recall that $H_0 \subset \mathbb{P}^n$ denotes the hypersurface at infinity. Without loss of generality, we may assume that $U \cap I^\pm = \emptyset$ and $U \cap V = \emptyset$. Furthermore, since $\overline{K^+} \cap H_0 = I^+$, we may choose U such that $\overline{U \setminus \hat{V}^+} \cap K^+ = \emptyset$. From $\hat{f}(H_0 \setminus I^+) = I^-$ and the uniform convergence of $f^k(p)$ to I^- on compact subsets of $\mathbb{C}^n \setminus K^+$, we conclude that there exists $N_1 > 0$ such that $\widehat{f^k}(U) \subset \hat{V}^-$ for $k \geq N_1$. In view of property (b) in (2), it follows that Claim 1 holds for any point $p \in (U \cup \hat{V}^+ \cup \hat{V}^-) \cap \mathbb{C}^n$ and all $N \geq N_1$. We note that $H_0 \subset U \cup \hat{V}^+ \cup \hat{V}^-$.

We consider now the set $D = \mathbb{C}^n \setminus (U \cup \hat{V}^+ \cup \hat{V}^- \cup V)$. Since $K \subset \operatorname{int} V$, it follows that $K^\pm$ is closed and, in view of (1), there exists an $\varepsilon > 0$ such that

$$K_\varepsilon^+ \cap K_\varepsilon^- \subset V, \tag{4}$$

where $K_\varepsilon^\pm$ denotes the ε-neighborhood of $K^\pm$. We define the compact set $D^\pm = \overline{D \setminus K_\varepsilon^\pm}$. Clearly $D^\pm \cap K^\pm = \emptyset$, and from (4) we have $D \subset D^+ \cup D^-$. From the uniform convergence of $f^k(p)$ to I^- on D^-, we conclude that there exists an $N_2 > 0$ such that $f^k(p) \subset \hat{V}^-$ for $k \geq N_2$ and $p \in D^-$. Analogously, $f^{-k}(p) \subset \hat{V}^+$ for $p \in D^+$ and for $k \geq N_3 > 0$. Combining all these considerations, we conclude that Claim 1 holds for $N = \max(N_1, N_2, N_3)$.

Claim 2. Let N be as in Claim 1. Then $V_N^- \cup V_N^+ \cup V = \mathbb{C}^n$.

We prove this claim by contradiction. We assume that there exists a point $p \in \mathbb{C}^n \setminus (V \cup V_N^+ \cup V_N^-)$. By Claim 1, either $f^N(p) \in \hat{V}^-$ or $f^{-N}(p) \in \hat{V}^+$. If $f^{-N}(p) \in \hat{V}^+ \cap \mathbb{C}^n = V_0^+$ then $p \in f^N(V_0^+) = V_N^+$, which is a contradiction. Now we suppose that $f^N(p) \in \hat{V}^-$. If $f^N(p) \in V_0^-(N)$ then $p \in f^{-N}(V_0^-(N)) = V_N^-$, which is again a contradiction. The remaining case is $f^N(p) \in (\hat{V}^- \cap \mathbb{C}^n) \setminus V_0^-(N)$. Since $f^N(p)$ is not contained in $V_0^-(N)$, it follows that $f^{N-j}(p) \in V_N^+$ for some $j \leq N$. But then the invariance of V_N^+ under f^{-1} implies that $p \in V_N^+$. This proves Claim 2.

We finally set $V^\pm = V_N^\pm$, where N is such that $\mathbb{C}^n \setminus V \subset V_N^+ \cup V_N^-$. We remark that V^- cannot be empty, because otherwise there would exist a neighborhood of I^- whose intersection with $\mathbb{C}^n$ is contained in V^+. Then, since the closure of K^- contains I^-, there exists a point $p \in K^-$ that is also contained in V^+. By construction, $f^{-N}(p) \in \hat{V}^+$. But $\hat{V}^+ \cap K^- = \emptyset$, and this contradicts the fact that K^- is an f-invariant set.

We redefine the set V by setting $V = \mathbb{C}^n \setminus (V^+ \cup V^-)$. This implies that V, V^+, V^- are pairwise disjoint. Moreover, V is compact. Indeed, the union of V^+ and V^- is not changed by the shrinking and hence $V^+ \cup V^-$ is open. We claim that $K \subset \operatorname{int} V$. To see this we note that, by continuity, for every $p \in K$ there exists a neighborhood $U \subset \mathbb{C}^n$ of p such that $f^N(U) \cap \hat{V}^- = \emptyset$ and $f^{-N}(U) \cap \hat{V}^+ = \emptyset$; this implies $U \subset V$.

It follows immediately from the construction that $f^{-1}(V^+) \subset V^+$, that is, inclusion (iii) holds. To obtain property (i), we first observe that it is sufficient to show $f(V_0^-(N)) \subset V_0^-(N)$. Let $p \in V_0^-(N)$, in particular, $p \notin f^N(V_N^+)$. Using the fact that $f^j(V_N^+) \subset f^{j+1}(V_N^+)$, we obtain $f(p) \notin f^{N+1}(V_N^+) \supset f^N(V_N^+)$, which implies $f(p) \in V_0^-(N)$ and so (i) holds. We now show property (ii). If $p \in V^-$, then $f(p) \in V^-$ by (i). Let now $p \in V$, and assume that $f(p) \in V^+$. It then follows from (iii) that $p \in V^+$, a contradiction. Analogously, we obtain property (iv). □

COROLLARY 3.2. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$, and let the compact set $V \subset \mathbb{C}^n$ be defined as in Proposition 3.1. Then*

$$\begin{aligned} f^{\pm 1}(K^\pm \cap V) &\subset K^\pm \cap V, \\ f^{\pm 1}(J^\pm \cap V) &\subset J^\pm \cap V. \end{aligned} \tag{5}$$

Proof. Suppose $p \in K^+ \cap V$. Then, by Proposition 3.1, $f(p) \in V$ or $f(p) \in V^-$. We need only consider the case $f(p) \in V^-$. It follows from the construction that $f^{N+1}(p) \in V_0^-(N) \subset \hat{V}^-$. On the other hand, K^+ is an f-invariant set and $K^+ \cap \hat{V}^- = \emptyset$. This is a contradiction. Similarly, one can easily verify the other cases. The proof of the second inclusion in (5) follows from the first inclusion and the f-invariance of $J^\pm$. □

REMARK. We note that, for $n = 2$ (i.e., when f is a finite composition of generalized Hénon mappings), V can be chosen to be a closed bidisk of a sufficiently large radius R; moreover, $V^- = \{(x, y) \in \mathbb{C}^2 : |y| > R \text{ and } |y| > |x|\}$ and $V^+ = \{(x, y) \in \mathbb{C}^2 : |x| > R \text{ and } |y| < |x|\}$ (see [BS1]).

For a set $X \subset \mathbb{C}^n$ we define the stable and unstable sets $W^s(X)$ and $W^u(X)$ as

$$\begin{aligned} W^s(X) &= \{q \in \mathbb{C}^n : \operatorname{dist}(f^k(q), f^k(X)) \to 0 \text{ as } k \to \infty\}, \\ W^u(X) &= \{q \in \mathbb{C}^n : \operatorname{dist}(f^{-k}(q), f^{-k}(X)) \to 0 \text{ as } k \to \infty\}. \end{aligned} \tag{6}$$

LEMMA 3.3. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$. Then the following statements hold*:

(i) $W^s(K) = K^+$;
(ii) $W^u(K) = K^-$;
(iii) $\bigcup f^k(V^+) = \mathbb{C}^n \setminus K^-$;
(iv) $\bigcup f^{-k}(V^-) = \mathbb{C}^n \setminus K^+$;
(v) *if* $|\det Df| = 1$, *then* $\operatorname{int} K^+ = \operatorname{int} K^- = \operatorname{int} K$;
(vi) *if* $|\det Df| < 1$, *then K^- has zero Lebesgue measure* (*in particular, we have* $\operatorname{int} K^- = \emptyset$);
(vii) $\{f^{\pm k} : k \in \mathbb{N}\}$ *is a normal family on* $\operatorname{int} K^\pm$.

Proof. The proof of inclusions (i) and (ii) is analogous to that in the case $n = 2$ (see [BS1]).

(iii) We first assume $p \in \mathbb{C}^n \setminus K^-$. Then $f^{-k}(p)$ converges to I^+ and so $f^{-k}(p) \in V_0^+ \subset V^+$ for sufficiently large k. On the other hand, if $p \in \bigcup f^k(V^+)$

then, by Proposition 3.1(i), there exists a $k_0 \in \mathbb{N}$ such that $f^{-k}(p) \in V^+$ for all $k \geq k_0$. Therefore, $f^{-k}(p)$ cannot converge to $K \subset \operatorname{int} V$. Hence $p \notin K^-$, and property (iii) holds.

(iv) Suppose $p \in \mathbb{C}^n \setminus K^+$. Then $f^k(p)$ converge to I^- as $k \to \infty$. Let us show that $\mathbb{C}^n \setminus K^+ \subset \bigcup f^{-k}(V^-)$. Suppose on the contrary that $f^k(p) \notin V^-$ for any $k > 0$. Then, since the orbit of p converges to I^-, there exists a $k_0 > 0$ such that $f^k(p) \in V^+$ for all $k > k_0$. Let N be as in Claim 1 of the proof of Proposition 3.1, and let $k > N$ be arbitrary. Then $f^k(p) \in V^+$ implies $f^{k-N}(p) \in V_0^+$; that is, all iterates of p stay in V_0^+. But $V_0^+ \cap \hat{V}^- = \emptyset$ and therefore, since $\hat{V}^-$ is a neighborhood of I^-, this contradicts the fact that $f^k(p)$ converge to I^-. The opposite inclusion can be proven similarly to case (iii).

(v) Assume on the contrary that there exists a ball $B = B(p,r) \subset \operatorname{int} K^+ \setminus \operatorname{int} K$. Without loss of generality we may assume $B \subset \operatorname{int} K^+ \setminus K$, in particular, $B \subset \mathbb{C}^n \setminus K^-$. Since f^{-k} converges uniformly to I^+ on compact subsets of $\mathbb{C}^n \setminus K^-$, there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ such that the sets $f^{-k_1}(B), f^{-k_2}(B), f^{-k_3}(B), \ldots$ are pairwise disjoint. Using that f^{-1} is volume-preserving, we obtain that $\operatorname{vol}(\operatorname{int} K^+) = \infty$ (here "vol" denotes the Lebesgue measure in $\mathbb{C}^n$). Thus there exists an $r > 0$ such that $\operatorname{vol}(B(0,r) \cap \operatorname{int} K^+) > \operatorname{vol}(V)$. By the uniform convergence of f^k on compact subsets of K^+, there exists a $k_0 \in \mathbb{N}$ such that $f^{k_0}(\operatorname{int} K^+ \cap B(0,r)) \subset V$. But this is a contradiction to $\operatorname{vol}(f^{k_0}(\operatorname{int} K^+ \cap B(0,r))) = \operatorname{vol}(\operatorname{int} K^+ \cap B(0,r)) > \operatorname{vol}(V)$. Thus $\operatorname{int} K^+ = \operatorname{int} K$ holds. The proof of the identity $\operatorname{int} K^- = \operatorname{int} K$ is analogous.

Property (vi) follows analogously to the case $n = 2$ (see [FM]). Finally, (vii) follows from [Si, Prop. 2.2.7]. □

4. Hyperbolicity

For generalized Hénon maps in $\mathbb{C}^2$, the concept of hyperbolicity was studied in detail by Bedford and Smillie (see [BS1]). Using the filtration properties obtained in Section 3, we generalize in this section some of the results of [BS1] to regular polynomial automorphisms of $\mathbb{C}^n$.

We first give some basic definitions and refer the reader to [KH] for details. Let f be a regular polynomial automorphism of $\mathbb{C}^n$. We say that a compact f-invariant set $\Lambda \subset \mathbb{C}^n$ is a *hyperbolic set* for f if there exists a continuous Df-invariant splitting of the tangent bundle $T_\Lambda \mathbb{C}^n = E^u \oplus E^s$ such that Df is uniformly expanding on E^u and uniformly contracting on E^s.

An important feature of hyperbolic sets is that we can associate with each point $p \in \Lambda$ its local unstable/stable manifold $W_\varepsilon^{u/s}(p)$. The local unstable/stable manifolds are complex manifolds of the same (complex) dimension as $E_p^{u/s}$. We denote the (global) unstable/stable manifolds at a point p (see (6)) by $W^{u/s}(p)$. It follows from the work of Jonsson and Varolin [JV] that there exists a Borel set $X \subset \Lambda$ with $\mu(X) = 1$ for every f-invariant Borel probability measure μ on Λ such that, for all $p \in X$, the global unstable/stable manifolds $W^{u/s}(p)$ are biholomorphic copies of $\mathbb{C}^k$, $k = \dim_{\mathbb{C}} E_p^{u/s}$. Such a set X is also called a set of full probability. We call $\dim_{\mathbb{C}} E_p^{u/s}$ the *unstable/stable index* of Λ at p. Note that the unstable/stable index is locally constant. If Λ is a hyperbolic set for f, we say that Λ is *locally maximal* if there exists a neighborhood U of Λ such that every hyperbolic set of f in U is contained in Λ. We say that an f-invariant set X has a *local product structure* if for all $p, q \in X$ we have $W^s(p) \cap W^u(q) \subset X$. We now consider the situation when J is a hyperbolic set for f.

PROPOSITION 4.1. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$, and suppose that J is a hyperbolic set for f. Then the following statements hold*:

(i) *if $p \in J$, then $W^{s/u}(p) \subset J^{\pm}$*;
(ii) *the set J has a local product structure*;
(iii) *the set J is locally maximal, and $W^{u/s}(J) = \bigcup_{p\in J} W^{u/s}(p)$*;
(iv) $W^{s/u}(J) \subset J^{\pm}$.

Proof. (i) Without loss of generality, we show the inclusion only for $W^s(p)$; the proof for $W^u(p)$ is analogous. Clearly, $W^s(p) \subset K^+$. Suppose there exists a point $q \in W^s(p) \cap \operatorname{int} K^+$. Then, since the family $\{f^k : k \in \mathbb{N}\}$ is normal in a neighborhood of q, the derivatives of f^k at q are bounded. On the other hand, by extending the hyperbolic structure of f to a neighborhood of J, it follows that the derivatives of f^k at q are unbounded.

(ii) If $p, q \in J$, then by (i) we have $W^s(p) \in J^+$ and $W^u(q) \in J^-$. Therefore, the intersection is in J.

(iii) The local product structure combined with hyperbolicity implies that J is locally maximal (see [Sh, Prop. 8.22]). The second statement is an application of the shadowing lemma for locally maximal hyperbolic sets (see [Bo]).

Finally, (iv) follows from (i) and (iii). □

Let C be a connected component of $\operatorname{int} K^+$. We say that C is *periodic* if there exists an $N \in \mathbb{N}$ such that $f^N(C) = C$. Otherwise we call C *wandering*. If α is a periodic point such that for all p in a neighborhood of α we have $f^k(p) \to f^k(\alpha)$ as $k \to \infty$, then we call α an *attracting periodic point*. Furthermore, $C = \{p \in \mathbb{C}^n : f^k(p) \to f^k(\alpha)\}$ is a periodic connected component of $\operatorname{int} K^+$ and is called the *basin of attraction* of α.

THEOREM 4.2. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$ with $|\det Df| \leq 1$. Suppose J is a hyperbolic set for f. Then the following hold*:

(i) *there are no wandering components in* $\operatorname{int} K^+$;
(ii) *each periodic component of* $\operatorname{int} K^+$ *is the basin of attraction of an attracting periodic point*;
(iii) *there are at most finitely many basins of attraction.*

The proof of Theorem 4.2 is analogous to that of Theorem 5.6 in [BS1]. We note that the references to Propositions 5.1 and 5.2 in the proof of [BS1] should be replaced by references to Proposition 4.1 stated here.

COROLLARY 4.3. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$. Assume that J is hyperbolic and that* $|\det Df| = 1$. *Then* $\operatorname{int} K^+ = \operatorname{int} K^- = \operatorname{int} K = \emptyset$.

Proof. By Theorem 4.2, the interior of K^+ is a finite union of basins of attraction; but since $|\det Df| = 1$, it is impossible to have a basin of attraction. Hence $\operatorname{int} K^+ = \emptyset$. Therefore, the corollary follows from Lemma 3.3(v). □

We need the following definitions. Let X be a topological space and let $T: X \to X$ be a continuous map. We call $x \in X$ a *nonwandering* point of T if, for every neighborhood U of x, there exists a $k \in \mathbb{N}$ such that $U \cap T^k(U) \neq \emptyset$. Otherwise we call x *wandering.* The set of nonwandering points of T is called the *nonwandering set* of T and is denoted by $\Omega(T)$.

We say that a regular polynomial automorphism f of $\mathbb{C}^n$ is *hyperbolic* if J is a hyperbolic set for f and the periodic points of $f|J$ are dense in J. We note that this definition of hyperbolicity is equivalent to J being hyperbolic and $\Omega(f|J) = J$ (see e.g. [KH]).

THEOREM 4.4. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$ with $|\det Df| \leq 1$. Then*:

(i) $W^s(J) = J^+$;
(ii) $W^u(J) = J^- \setminus \{\alpha_1, \dots, \alpha_m\}$, *where the α_i are the attracting periodic points of f*;
(iii) $\Omega(f) = J \cup \{\alpha_1, \dots, \alpha_m\}$.

Proof. (i) To prove $W^s(J) = J^+$ we first observe that, by Proposition 4.1(iv), $W^s(J) \subset J^+$. To show the reverse inclusion we notice that Lemma 3.3(i) implies that $J^+ \subset W^s(K)$. If $p \in J^+$ then the iterates of p converge to $K \cap J^+ = K^- \cap J^+$. To prove (i) we claim that $K^- \cap J^+ = J^- \cap J^+ = J$. In the case $|\det Df| = 1$, the claim follows from Lemma 3.3(v). For $|\det Df| < 1$, the claim follows from Lemma 3.3(vi).

(ii) Obviously every attracting period point belongs to $\operatorname{int} K^+$ and so $W^u(J) \subset J^- \setminus \{\alpha_1, \dots, \alpha_m\}$ follows from Proposition 4.1(iv). In order to show the opposite inclusion we consider $p \in J^- \setminus \{\alpha_1, \dots, \alpha_m\}$. If $p \in \mathbb{C}^n \setminus K^+$, then the backward orbit of p converges to $K \cap \partial(\mathbb{C}^n \setminus K^+) = K \cap J^+$. Using the fact that J^- is a closed invariant set, we conclude that the backward orbit of p must converge to $J^+ \cap J^- = J$. If $p \in J^+$, then $p \in J$ and there is nothing to prove. To complete the proof of (ii) we must consider the case $p \in \operatorname{int} K^+$. By Theorem 4.2, there exists an attracting periodic point α_i such that p is contained in the basin of attraction C of α_i. Without loss of generality we assume that α_i is an attracting fixed point (i.e., that C is an f-invariant component). Let V be as in Proposition 3.1, and let $U \subset V \cap C$ be an open neighborhood of α_i such that $f(\bar{U}) \subset U$. Such a set U always exists because α_i is an attracting fixed point. Obviously, $\bigcup f^{-k}(U)$ is an exhaustion of C. Thus $\bigcup f^{-k}(U \cap J^-)$ is an exhaustion of $C \cap J^-$. Therefore, since $p \in K$, we may conclude that the backward orbit of p cannot have a cluster point in C and hence must converge to $\partial C \subset J^+$. Thus $p \in J^- \setminus \{\alpha_1, \dots, \alpha_m\}$ implies $p \in W^u(J)$.

(iii) Evidently every periodic point of f belongs to the nonwandering set of f. Since the nonwandering set is closed, it follows that $\Omega(f) \supset J \cup \{\alpha_1, \dots, \alpha_m\}$.

Let $p \in \mathbb{C}^n$ be a nonwandering point for f. If p is not an attracting periodic orbit then p can not belong to $\operatorname{int} K^+$, since otherwise Theorem 4.2 would imply that its forward orbit converges to an attracting periodic orbit. On the other hand $p \notin \mathbb{C}^n \setminus K^+$, because in this case the forward orbit of p would converge to I^-. Hence $p \in J^+$, and it follows from (i) that $p \notin J^+ \setminus J$. This implies $p \in J$, which completes the proof. □

We say that a diffeomorphism on a Riemannian manifold is *Axiom A* if its nonwandering set is a hyperbolic set and the periodic points are dense in the nonwandering set.

COROLLARY 4.5. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$. Then f is Axiom A.*

Proof. This is a consequence of Theorem 4.4(iii) and the fact that every attracting periodic point is an isolated hyperbolic set. □

REMARKS. 1. As noted in Section 1, if $n = 2$ and J is hyperbolic then the periodic points are dense in J (see [BS1]). This provides a weaker definition of hyperbolic maps. We do not know whether the analogous result holds in the case $n > 2$. However, there exist examples of diffeomorphisms of higher-dimensional real manifolds with the property that the nonwandering set is a hyperbolic set, but the periodic points are not dense in it (see [Da]).

2. It is shown in [MNTU, Thm. 9.3.14] that, when $n = 2$, hyperbolicity is equivalent to Axiom A. We do not know whether the analogous result holds for $n > 2$. The difficulty is proving that the Julia set of an Axiom A regular polynomial map is a subset of the nonwandering set. This is shown in the case $n = 2$ using methods that are not available in higher dimensions.

5. Dimension Theory

In this section we study the Hausdorff dimension and box dimension of the Julia sets of a regular polynomial automorphism of $\mathbb{C}^n$.

We first prove a result about the entropy of f that is also of independent interest. Namely, we show that the Julia set carries the full entropy of f. We then apply this result to study the dimensions of the Julia sets.

For a continuous map T on a compact metric space X, we denote by $h_{\mathrm{top}}(T)$ the topological entropy of T (see [Wa] for details).

THEOREM 5.1. *Let f be regular polynomial automorphism of $\mathbb{C}^n$, and assume that I^- has dimension $l-1$. Then $h_{\mathrm{top}}(f|J) = l \log d$.*

Proof. Without loss of generality we assume $|\det Df| \leq 1$. We claim that $K \setminus J \subset \operatorname{int} K^+$. Indeed, we have $K = (\operatorname{int} K^+ \cup J^+) \cap (\operatorname{int} K^- \cup J^-)$. This implies

$$K \setminus J \subset \operatorname{int} K^+ \cup \operatorname{int} K^-. \tag{7}$$

If $|\det Df| < 1$ then, by Lemma 3.3(vi), $\operatorname{int} K^- = \emptyset$. Therefore, (7) implies $K \setminus J \subset \operatorname{int} K^+$. On the other hand, if $|\det Df| = 1$ then by Lemma 3.3(v) we have $\operatorname{int} K^+ = \operatorname{int} K^- = \operatorname{int} K$. Again by (7) we obtain $K \setminus J \subset \operatorname{int} K^+$.

Sibony observed in [Si] that $h_{\mathrm{top}}(f|K) = l \log d$. Let $\delta > 0$. It follows from the variational principle (see e.g. [Wa]) that there exists an f-invariant probability measure μ on K with $h_\mu(f|K) > l \log d - \delta$, where $h_\mu(f|K)$ denotes the measure-theoretic entropy of $f|K$ with respect to μ. Let τ be an ergodic decomposition of μ. This means that τ is a probability measure on the metrizable space $\mathcal{M}$ of f-invariant probability measures on K that puts full measure on the subset $\mathcal{M}_E$ of ergodic measures. Furthermore,

$$\int_{\mathcal{M}} \int_K \varphi \, d\nu \, d\tau(\nu) = \int_K \varphi \, d\mu \tag{8}$$

for every $\varphi \in C(K, \mathbb{R})$. Since

$$h_\mu(f|K) = \int_{\mathcal{M}} h_\nu(f|K) \, d\tau(\nu), \tag{9}$$

there exists a $\nu \in \mathcal{M}_E$ such that $h_\nu(f|K) > l \log d - \delta$.

Next we claim that $\nu(J) = 1$. If not, then $\nu(K \setminus J) > 0$. Since ν is ergodic, this would imply $\nu(K \setminus J) = 1$ and $\nu(J) = 0$. By work of Brin and Katok [BK], for ν-a.e. $p \in K$ (and thus in particular for ν-a.e. $p \in K \setminus J$) there exists the limit

$$h_\nu(p) = \lim_{\varepsilon \to 0} \lim_{k \to \infty} -\frac{1}{k} \log \nu(B(p, \varepsilon, k)), \tag{10}$$

where

$$B(p, \varepsilon, k) = \{q \in \mathbb{C}^n : |f^i(q) - f^i(p)| < \varepsilon \text{ for } i = 1, \dots, k-1\}. \tag{11}$$

The number $h_\nu(p)$ (if it exists) is called the *local entropy* of ν at p. Furthermore, the function $p \mapsto h_\nu(p)$ is ν integrable, and we have

$$h_\nu(f|K) = \int_K h_\nu(p) \, d\nu = \int_{K \setminus J} h_\nu(p) \, d\nu. \tag{12}$$

The right-hand equality in (12) follows from $\nu(J) = 0$. Let now $p \in K \setminus J$ be such that the limit in (10) exists. In particular, $p \in \operatorname{supp} \nu$. Choose $\varepsilon > 0$ such that $\overline{B(p, \varepsilon)} \subset \operatorname{int} K^+$. It follows from Lemma 3.3(vii) that the derivatives of f^k ($k \in \mathbb{N}$) are bounded on $\overline{B(p, \varepsilon)}$. Applying the mean value theorem yields the existence of $\varepsilon' > 0$ such that $B(p, \varepsilon') \subset B(p, \varepsilon, k)$ for all $k \in \mathbb{N}$. Hence $\nu(B(p, \varepsilon, k)) \geq \nu(B(p, \varepsilon')) > 0$ for all $k \in \mathbb{N}$. Therefore, (10) implies $h_\nu(p) = 0$ and by (12) we obtain $h_\nu(f|K) = 0$. This is a contradiction, so we must have $\nu(J) = 1$, which implies that $h_\nu(f) = h_{\nu|J}(f)$. Finally, since δ can be chosen arbitrarily small, the variational principle yields $h_{\mathrm{top}}(f|J) = l \log d$. $\square$

The following result is a consequence of the proof of Theorem 5.1.

COROLLARY 5.2. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$. Then* $h_{\mathrm{top}}(f|K \setminus J) = 0$.

We note that, in Corollary 5.2, h_{top} denotes the entropy for maps on noncompact spaces (see [Wa]).

5.1. *Dimension for a General Map*

We now consider the dimensions of the Julia sets without the assumption of hyperbolicity.

Let f be a regular polynomial automorphism of $\mathbb{C}^n$ and let $V \subset \mathbb{C}^n$ be a compact set with $K \subset \operatorname{int} V$ and $f^{\pm 1}(J^{\pm} \cap V) \subset J^{\pm} \cap V$ (see Corollary 3.2). We define

$$s_V^{\pm} = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\max\{\|Df^{\pm k}(p)\| : p \in J^{\pm} \cap V\}\bigr). \tag{13}$$

The submultiplicativity of the operator norm guarantees the existence of the limit defining $s_V^{\pm}$. Since all norms in $\mathbb{C}^n$ are equivalent, the value of $s_V^{\pm}$ is independent of the norm. It follows from Theorem 5.1 (also using the variational principle) that there exists an f-invariant probability measure on J with positive measure-theoretic entropy. Therefore, Ruelle's inequality implies that $s_V^{\pm}$ is strictly positive.

LEMMA 5.3. *The value of $s_V^{\pm}$ is independent of the choice of V.*

Proof. It is shown in Lemma 3.3 that $W^s(K) = K^+$ and $W^u(K) = K^-$. Therefore, the proof follows by a standard argument. □

In view of Lemma 5.3 we set $s^{\pm} = s_V^{\pm}$. Given a set $A \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, we denote by $\dim_H A$ the Hausdorff dimension of A and (provided A is bounded) by $\overline{\dim}_B A$ its upper box dimension (see [Ma] for details). Then $\dim_H A \le \overline{\dim}_B A$ holds for an arbitrary set A, while the equality holds if A is a sufficiently regular set. We now consider the volume-decreasing case (i.e., when $|\det Df| < 1$). The following theorem provides an upper bound for the dimension of K^-.

THEOREM 5.4. *Let f be a volume-decreasing regular polynomial automorphism of $\mathbb{C}^n$. Assume that V is as in Proposition 3.1. Then*

$$\overline{\dim}_B K^- \cap V \le 2n + \frac{2\log|\det Df|}{s^-} < 2n. \tag{14}$$

Proof. By Lemma 3.3(vi), $K^- = J^-$. Therefore, it is sufficient to show inequality (14) for $J^- \cap V$. Note that the real Jacobian of f^{-1} as a map of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ is equal to $|\det Df|^{-2}$. The result now follows immediately from [Wo2, Thm. 1.1]. □

REMARK. Since $W^u(K) = K^-$, we can define an exhaustion $V_k = f^k(V \cap K^-)$ of K^-. This implies that the upper bound in inequality (14) provides also an upper bound for the Hausdorff dimension of K^-.

COROLLARY 5.5. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$ that is not volume-preserving. Then $\overline{\dim}_B K < 2n$.*

Proof. If f is not volume-preserving, then either f or f^{-1} is volume-decreasing. The result now follows immediately from Theorem 5.4. □

REMARK. It should be noted that Corollary 5.5 does not hold without the assumption that f is not volume-preserving. In fact, there exist volume-preserving regular polynomial automorphisms with a Siegel ball, in which case K has a nonempty interior.

As consequence of Theorem 5.1, we obtain that the upper box dimension of J is strictly positive.

COROLLARY 5.6. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$, and assume that I^- has dimension $l-1$. Define*

$$s^{\pm} = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\max\{\|Df^{\pm k}(p)\| : p \in J\}\bigr).$$

Then

$$\overline{\dim}_B J \geq \max\left\{\frac{l \log d}{s^+}, \frac{l \log d}{s^-}\right\}. \tag{15}$$

In particular, $\overline{\dim}_B J > 0$.

Proof. We have $h_{\mathrm{top}}(f|J) = h_{\mathrm{top}}(f^{-1}|J) = l \log d$. Therefore, inequality (15) follows from [KH, Thm. 3.2.9] and a standard limit argument. Finally, $s^{\pm} > 0$ follows analogously as the positivity of $s_V^{\pm}$ (see (13)), and therefore $\overline{\dim}_B J > 0$. $\square$

Next we give a lower bound for the Hausdorff dimension of J^+ in the case when I^- has dimension zero. For this we introduce the positive Green function

$$G^+(p) = \lim_{k\to\infty} \frac{1}{d^k} \log^+|f^k(p)|. \tag{16}$$

We note that G^+ is a well-defined Hölder-continuous function with $K^+ = \{G^+ = 0\}$ (see [Si] for details).

PROPOSITION 5.7. *Let f be a regular polynomial automorphism of $\mathbb{C}^n$. Assume that I^- has dimension zero and let s^+ be as in* (13). *Then, for all $s_0^+ > s^+$, the positive Green function G^+ is Hölder-continuous on compact subsets of $\mathbb{C}^n$ with Hölder exponent* $(\log d)/s_0^+$. *Furthermore,*

$$\dim_H J^+ \geq 2n - 2 + \frac{\log d}{s^+} > 2n - 2. \tag{17}$$

Proof. Let $s_0^+ > s^+$. Using the filtration properties (see Proposition 3.1), one can show the Hölder continuity of G^+ on compact subsets of $\mathbb{C}^n$ with Hölder exponent $(\log d)/s_0^+$ using arguments analogous to those used by Fornæss and Sibony [FS] in the case $n = 2$ (see also [Si]). By [Si, Prop. 2.2.10], the positive Green function G^+ is pluriharmonic on $\mathbb{C}^n \setminus J^+$. Furthermore, the maximum principle for pluriharmonic functions implies that G^+ cannot be extended as a pluriharmonic map to any neighborhood of any point of J^+. Inequality (17) now follows by a classical result of Carleson about the Hausdorff dimension of removable sets for Hölder-continuous harmonic functions (see [Ca]). $\square$

REMARKS. (i) The analogous result holds for the Hausdorff dimension of J^- if I^+ is zero-dimensional.

(ii) We note that Proposition 5.7 is of interest only if int $K^+ = \emptyset$, since otherwise J^+ has topological dimension $2n-1$.

5.2. Dimension for Hyperbolic Maps

We now consider hyperbolic maps. It is well known that a locally maximal hyperbolic set, which carries positive topological entropy, has positive Hausdorff dimension. For a hyperbolic regular automorphism f of $\mathbb{C}^n$, the positivity of the Hausdorff dimension of the Julia set can be shown (for instance) by the following argument.

Let $J = J_1 \cup \cdots \cup J_m$ be the decomposition of J into basic sets (see e.g. [Bo] for details); we note that, when $n = 2$, the Julia set J is the unique basic set of f that is not an attracting periodic orbit (see [BS1]). Since $h_{\text{top}}(f|J) = l \log d$ (see Theorem 5.1), there exists an $i \in \{1, \dots, m\}$ such that $h_{\text{top}}(f|J_i) = l \log d$. Here $l = \dim I^- + 1$. It is a well-known fact that there exists a unique f-invariant probability measure μ_i of maximal entropy for $f|J_i$ (see e.g. [KH]). Moreover, μ_i is ergodic. We define

$$s_i^{\pm} = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\max\{\|Df^{\pm k}(p)\| : p \in J_i\}\bigr). \tag{18}$$

Applying [Y, Cor. 5.1] then yields

$$l \log d\left(\frac{1}{s_i^+} + \frac{1}{s_i^-}\right) \le \dim_H \mu_i \le \dim_H J_i \le \dim_H J, \tag{19}$$

where $\dim_H \mu_i = \inf\{\dim_H A : \mu_i(A) = 1\}$ denotes the Hausdorff dimension of the measure μ_i. Note that, in general, the Hausdorff dimension of an f-invariant measure provides only a rough estimate of the Hausdorff dimension of the Julia set. In fact, it was shown in [Wo3] that, for a generic hyperbolic polynomial automorphism f of $\mathbb{C}^2$, there exists an $\varepsilon > 0$ (which depends on f) such that $\dim_H \nu < \dim_H J - \varepsilon$ for all ergodic f-invariant probability measures ν.

Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$, let $J_i \subset J$ be a basic set of f, and let $\varphi \in C(J_i, \mathbb{R})$. We denote by $P(f|J_i, \varphi)$ the topological pressure of φ with respect to $f|J_i$ (see [KH] for the definition and details). We consider the function $\phi^u = -\log\|Df|E^u\|$. Note that ϕ^u is Hölder-continuous; see [Bo].

We now consider the case when the unstable index J_i is identically 1.

THEOREM 5.8. *Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be a hyperbolic regular polynomial automorphism and let $J_i \subset J$ be a basic set of f. Assume that the unstable index of J_i is identically* 1. *Then $t^u = \dim_H W_\varepsilon^u(p) \cap J_i$ is independent of $p \in J_i$ and* $0 < t^u < 2$. *Moreover, t^u is given by the unique solution of*

$$P(f|J_i, t\phi^u) = 0. \tag{20}$$

Equation (20) is usually called the Bowen–Ruelle formula. We refer to t^u as the Hausdorff dimension of the unstable slice. Theorem 5.8 is a special case of [Pe, Thm. 22.1]. In the case $n = 2$, Theorem 5.8 is due to Verjovsky and Wu [VW]. We note that in this situation the stable index of the basic set J is also identically 1, and we obtain the analogous result to Theorem 5.8 for the Hausdorff dimension of the stable slice $t^s = \dim_H W^s_\varepsilon(p) \cap J$. Moreover, $\dim_H J = t^u + t^s$ (see [Wo1] and the references therein).

The following result is a version of [Wo1, Thm. 4.1]. The proof is analogous.

Theorem 5.9. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$ and let $J_i \subset J$ be a basic set of f. Assume that the unstable index of J_i is identically 1. Then $\dim_H W^s(J_i) = t^u + 2n - 2$. In particular, $2n - 2 < \dim_H W^s(J_i) < 2n$.*

Let now $A \subset \mathbb{C}^k$ be an open set and let $\{f_a : a \in A\}$ be a holomorphic family of hyperbolic regular polynomial automorphisms of $\mathbb{C}^n$. Let $a_0 \in A$ and let $J_{a_0,i} \subset J_{a_0}$ be a basic set of f_{a_0}. Let $U \subset \mathbb{C}^n$ be a neighborhood of $J_{a_0,i}$ with the property that, for all $a \in A$ close enough to a_0, f_a has a basic set $J_{a,i} \subset U$ such that $f_{a_0}|J_{a_0,i}$ is conjugate to $f_a|J_{a,i}$. For $p \in J_{a,i}$ we denote by t^u_a the Hausdorff dimension of $W^u_\varepsilon(p) \cap J_{a,i}$. Recall that, by Theorem 5.8, t^u_a does not depend on p. The following result can be proven analogously to the corresponding results in the case $n = 2$ (see [VW; Wo1]).

Theorem 5.10. *Assume that the unstable index of $J_{a,i}$ is identically 1 in a neighborhood of $a_0 \in A$. Then the functions $a \mapsto t^u_a$ and $a \mapsto \dim_H W^s(J_{a,i})$ are real-analytic and plurisubharmonic in a neighborhood of $a_0 \in A$.*

Remark. The corresponding versions of Theorems 5.8, 5.9, and 5.10 for the stable slices also hold—provided that the stable index is identically 1.

Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$ and let $J_i \subset J$ be a basic set of f. We define $\varphi^{u/s} : J_i \to \mathbb{R}$ by $\varphi^{u/s}(p) = \mp\log|\lambda(p)|$, where $\lambda(p)$ denotes the Jacobian of the linear map $Df^{\pm 1}(p)|E^{u/s}_p$. The following theorem is the main result of this section.

Theorem 5.11. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$, and let $J_i \subset J$ be a basic set of f. Define*

$$W^{s/u}_\varepsilon(J_i) = \bigcup_{p \in J_i} W^{s/u}_\varepsilon(p),$$

$$s^\pm = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\max\{\|Df^{\pm k}(p)\| : p \in J_i\}\bigr).$$

Then

$$\overline{\dim}_B W^{s/u}_\varepsilon(J_i) \le 2n + \frac{P(f|J_i, \varphi^{u/s})}{s^\pm} < 2n. \tag{21}$$

Proof. We prove the result only for $W^s_\varepsilon(J_i)$, as the proof for $W^u_\varepsilon(J_i)$ is entirely analogous. Since $J_i \subset J$, its unstable index must be at least 1, which implies

$s^+ > 0$. By Proposition 4.1, $W^s_\varepsilon(J_i) \subset J^+$; in particular, $W^s_\varepsilon(J_i)$ is not a neighborhood of J_i. Therefore, we may conclude from [Bo, Prop. 3.10, Prop. 4.8, Thm. 4.11] that $P(f|J_i, \varphi^u) < 0$. This gives the inequality on the right.

Let $\delta > 0$. It follows by a simple continuity argument that there exist $\varepsilon > 0$ and $k_\delta \in \mathbb{N}$ such that for all $p \in B(W^s_\varepsilon(J_i), \varepsilon) = \{p \in \mathbb{C}^n : \exists q \in W^s_\varepsilon(J_i), |p-q| < \varepsilon\}$ we have

$$\|Df^{k_\delta}(p)\| < \exp(k_\delta(s^+ + \delta)). \tag{22}$$

From now on we consider the map $g = f^{k_\delta}$. Note that J_i is also a basic set of g. Evidently $W^s_\varepsilon(J_i)$ is forward-invariant under g. It follows from the variational principle that $P(g|J_i, \varphi^u) = k_\delta P(f|J_i, \varphi^u)$; moreover, $s^+_g = k_\delta s^+_f$. It is thus sufficient to prove the left-hand side of inequality (21) for g. Let $p \in J_i$ and $k \in \mathbb{N}$. We recall that

$$B(p, \varepsilon, k) = \{q \in \mathbb{C}^n : |g^i(p) - g^i(q)| < \varepsilon,\ i = 0, \ldots, k-1\} \tag{23}$$

(see (11)), and we define $B(J_i, \varepsilon, k) = \bigcup_{p \in J_i} B(p, \varepsilon, k)$. Making ε smaller if necessary, it follows from [Bo, Prop. 4.8] that

$$P(g|J_i, \varphi^u) = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\mathrm{vol}(B(J_i, 2\varepsilon, k))\bigr). \tag{24}$$

For simplicity we write $b = P(g|J_i, \varphi^u)$. From (24) we obtain that if k is sufficiently large then

$$\mathrm{vol}(B(J_i, 2\varepsilon, k)) < \exp(k(b + \delta)). \tag{25}$$

For all $k \in \mathbb{N}$, we define real numbers

$$r_k = \frac{\varepsilon}{\exp(s^+ + \delta)^k} \tag{26}$$

and neighborhoods $B_k = B(W^s_\varepsilon(J_i), r_k)$ of $W^s_\varepsilon(J_i)$. Let $q \in B_k$. Then there exists a $p \in W^s_\varepsilon(J_i)$ with $|p-q| < r_k$. An elementary induction argument in combination with the mean-value theorem implies $|g^i(p) - g^i(q)| < \varepsilon$ for all $i \in \{0, \ldots, k-1\}$. Since p is contained in the local stable manifold of size ε of a point in J_i, it follows that $q \in B(J_i, 2\varepsilon, k)$. Hence $B_k \subset B(J_i, 2\varepsilon, k)$. Therefore, (25) implies

$$\mathrm{vol}(B_k) < \exp(k(b + \delta)) \tag{27}$$

for sufficiently large k. Let us recall that, for $t \in [0, 2n]$, the t-dimensional upper Minkowski content of a bounded set $A \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ is defined by

$$M^{*t}(A) = \limsup_{\rho\to 0} \frac{\mathrm{vol}(A_\rho)}{(2\rho)^{2n-t}}, \tag{28}$$

where $A_\rho = \{p \in \mathbb{C}^n : \exists q \in A : |p-q| \le \rho\}$. Let $t \in [0, 2n]$ and $\rho_k = r_k/2$ for all $k \in \mathbb{N}$. Then we have

$$
\begin{aligned}
M^{*t}(W_\varepsilon^s(J_i)) &= \limsup_{\rho\to 0} \frac{\mathrm{vol}(W_\varepsilon^s(J_i)_\rho)}{(2\rho)^{2n-t}} \\
&\le \limsup_{k\to\infty} \frac{\mathrm{vol}(W_\varepsilon^s(J_i)_{\rho_k})}{(2\rho_{k+1})^{2n-t}} \\
&\le \limsup_{k\to\infty} \frac{\mathrm{vol}(B_k)}{(r_{k+1})^{2n-t}} \\
&\le \frac{\exp(s^+ + \delta)^{2n-t}}{\varepsilon^{2n-t}} \lim_{k\to\infty} (\exp(s^+ + \delta)^{2n-t} \exp(b+\delta))^k. \qquad (29)
\end{aligned}
$$

Let $t > 2n + (b+\delta)/(s^+ + \delta)$. Then $\exp(s^+ + \delta)^{2n-t}\exp(b+\delta) < 1$. This implies that $M^{*t}(W_\varepsilon^s(J_i)) = 0$; in particular, $t \ge \overline{\dim}_B W_\varepsilon^s(J_i)$ (see [Ma]). Since δ can be chosen arbitrarily small, the result follows. □

Corollary 5.12. *Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be a hyperbolic regular polynomial automorphism. Then $\overline{\dim}_B J < 2n$.*

Proof. By the spectral decomposition, J is the union of finitely many basic sets. By Theorem 5.11, each of these basic sets has upper box dimension strictly smaller than $2n$, and the result follows. □

Remark. We note that Corollary 5.12 is of interest only if f is volume-preserving, because otherwise $\overline{\dim}_B J < 2n$ holds even without the assumption of hyperbolicity (see Corollary 5.5).

Theorem 5.13. *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$, and let $J_1, \dots, J_m$ be the basic sets of f that are contained in J. For each $i \in \{1, \dots, m\}$, define*

$$s_i^\pm = \lim_{k\to\infty} \frac{1}{k} \log\bigl(\max\{\|Df^{\pm k}(p)\| : p \in J_i\}\bigr)$$

and $b_i^\pm = P(f|J_i, \varphi^{u/s})$. Then

$$\dim_H J^\pm \le 2n + \max\left\{\frac{b_i^\pm}{s_i^\pm}\right\} < 2n. \qquad (30)$$

Proof. Without loss of generality we show the result only for J^+. By Theorem 5.11 we have

$$\overline{\dim}_B\Bigl(\bigcup_{p\in J_i} W_\varepsilon^s(p)\Bigr) \le 2n + \frac{b_i^+}{s_i^+} < 2n \qquad (31)$$

for all $i = 1, \dots, m$. This implies

$$\overline{\dim}_B\Bigl(\bigcup_{i=1}^m \bigcup_{p\in J_i} W_\varepsilon^s(p)\Bigr) \le 2n + \max\left\{\frac{b_i^+}{s_i^+}\right\} < 2n. \qquad (32)$$

Hence

$$\dim_H\left(\bigcup_{p\in J} W^s_\varepsilon(p)\right) \le 2n + \max\left\{\frac{b_i^+}{s_i^+}\right\} < 2n. \tag{33}$$

Because

$$W^s(p) = \bigcup_{k\in\mathbb{N}} f^{-k}(W^s_\varepsilon(f^k(p))) \tag{34}$$

for all $p \in J$, inequality (30) follows from Theorem 4.4(i), Proposition 4.1(iii), and the fact that the Hausdorff dimension is countable union stable. □

REMARK. In the case $n = 2$, the result $\dim_H J^{\pm} < 4$ was already shown in [Wo2]. However, the methods used in [Wo2] crucially depend on the fact that the unstable/stable index of J is identically 1, and hence they do not apply to the case $n > 2$.

References

[BP] E. Bedford and V. Pambuccian, *Dynamics of shift-like polynomial diffeomorphisms of* $\mathbb{C}^n$, Conform. Geom. Dyn. 2 (1998), 45–55.

[BS1] E. Bedford and J. Smillie, *Polynomial diffeomorphisms of* $\mathbb{C}^2$*: Currents, equilibrium measure and hyperbolicity,* Invent. Math. 103 (1991), 69–99.

[BS2] ———, *Polynomial diffeomorphisms of* $\mathbb{C}^2$ *2: Stable manifolds and recurrence,* J. Amer. Math. Soc. 4 (1991), 657–679.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Math., 470, Springer-Verlag, Berlin, 1975.

[BK] M. Brin and A. Katok, *On local entropy,* Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., 1007, pp. 30–38, Springer-Verlag, Berlin, 1983.

[Ca] L. Carleson, *Removable singularities of continuous harmonic functions in* $\mathbb{R}^m$, Math. Scand. 12 (1963), 15–18.

[CF] D. Coman and J. E. Fornæss, *Green's functions for irregular quadratic polynomial automorphisms of* $\mathbb{C}^3$, Michigan Math. J. 46 (1999), 419–459.

[Da] A. Dankner, *On Smale's Axiom A dynamical systems,* Ann. of Math. (2) 107 (1978), 517–553.

[FS] J. E. Fornæss and N. Sibony, *Complex Hénon mappings in* $\mathbb{C}^2$ *and Fatou–Bieberbach domains,* Duke Math. J. 65 (1992), 345–380.

[FW] J. E. Fornæss and H. Wu, *Classification of degree 2 polynomial automorphisms of* $\mathbb{C}^3$, Publ. Mat. 42 (1998), 195–210.

[FM] S. Friedland and J. Milnor, *Dynamical properties of plane polynomial automorphisms,* Ergodic Theory Dynam. Systems 9 (1989), 67–99.

[JV] M. Jonsson and D. Varolin, *Stable manifolds of holomorphic diffeomorphisms,* Invent. Math. 149 (2002), 409–430.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems,* Encyclopedia Math. Appl., 54, Cambridge Univ. Press, Cambridge, U.K., 1995.

[Ma] P. Mattila, *Geometry of sets and measures in Euclidean spaces,* Cambridge Stud. Adv. Math., 44, Cambridge Univ. Press, Cambridge, U.K., 1995.

[MNTU] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic dynamics,* Cambridge Stud. Adv. Math., 66, Cambridge Univ. Press, Cambridge, U.K., 1999.

[Pe] Ya. Pesin, *Dimension theory in dynamical systems: Contemporary views and applications,* Chicago Lectures in Math., Univ. Chicago Press, 1997.

[Sh] M. Shub, *Global stability of dynamical systems,* Springer-Verlag, New York, 1987.

[Si] N. Sibony, *Dynamique des applications rationnelles de* $\mathbb{P}^k$, Dynamique et geometrie complexes (Lyon, 1997), pp. 97–185, Soc. Math. France, Paris, 1999.

[Ur] M. Urbanski, *Measures and dimensions in conformal dynamics,* Bull. Amer. Math. Soc. (N.S.) 40 (2003), 281–321.

[VW] A. Verjovsky and H. Wu, *Hausdorff dimension of Julia sets of complex Hénon mappings,* Ergodic Theory Dynam. Systems 16 (1996), 849–861.

[Wa] P. Walters, *An introduction to ergodic theory,* Grad. Texts in Math., 79, Springer-Verlag, Berlin, 1982.

[Wo1] C. Wolf, *Dimension of Julia sets of polynomial automorphisms of* $\mathbb{C}^2$, Michigan Math. J. 47 (2000), 585–600.

[Wo2] ———, *On the box-dimension of an invariant set*, Nonlinearity 14 (2001), 73–79.

[Wo3] ———, *On measures of maximal and full dimension for polynomial automorphisms of* $\mathbb{C}^2$, Trans. Amer. Math. Soc. 355 (2003), 3227–3239.

[Y] L.-S. Young, *Dimension, entropy and Lyapunov exponents,* Ergodic Theory Dynam. Systems 2 (1982), 109–124.

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