REAL ANALYTIC SETS IN COMPLEX SPACES AND CR MAPS

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Abstract. If R is a real analytic set in $\mathbb{C}^n$ (viewed as $\mathbb{R}^{2n}$), then for any point $p \in R$ there is a uniquely defined germ X_p of the smallest complex analytic variety which contains R_p, the germ of R at p. It is shown that if R is irreducible of constant dimension, then the function $p \to \dim X_p$ is constant on a dense open subset of R. As an application it is proved that a continuous map from a real analytic CR manifold M into $\mathbb{C}^N$ which is CR on some open subset of M and whose graph is a real analytic set in $M \times \mathbb{C}^N$ is necessarily CR everywhere on M.

1. Introduction

Given a real analytic set R in $\mathbb{C}^n$ (we may identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$), $n > 1$, we consider the germ R_p of R at a point $p \in R$ and define X_p to be the germ at p of the smallest (with respect to inclusion) complex analytic set in $\mathbb{C}^n$ which contains R_p. Then X_p exists and unique for each $p \in R$, with $\mathbb{C}^n$ being such a set for a *generic* R. It is natural to ask how the dimension of X_p varies with $p \in R$. Consider the following example (Cartan's umbrella, see [3])

$$R = \left\{(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : x_2(x_1^2 + y_1^2) - x_1^3 = 0;\ y_2 = 0\right\}. \tag{1}$$

At a point $p = (0, x_2) \in \mathbb{C}^2$, $x_2 \neq 0$, the complex analytic set $X_p = \{z_1 = 0\}$ contains R_p, but at the origin $X_0 = \mathbb{C}^2$.

The set in (1) is irreducible but has different dimension at different points. In particular this set is not *coherent* (see Section 2 for definitions). Our main result is that under the assumption that R is irreducible and has constant dimension, the dimension of X_p is constant on a dense open subset of R. More precisely, the following result holds.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$, $n > 1$, and R be an irreducible real analytic subset of Ω of constant positive dimension. Then*

(i) There exists an integer $d > 0$ and a closed nowhere dense subset S of R (possibly empty) such that for any point $p \in R \setminus S$, $\dim X_p = d$.

(ii) If R is in addition coherent, then there exists a complex analytic set X, defined in a neighbourhood of $R \setminus S$ such that for any point $p \in R \setminus S$, the germ X_p is the smallest complex analytic set which contains the germ R_p.

The function $d(p) = \dim X_p$ is upper semicontinuous on R, and therefore, $\dim X_p \geq d$ for $p \in S$. We do not know any examples where R has constant dimension and S is nonempty, and it would be interesting to obtain further information about this set. The set S is empty in the case when the germ of R is complex analytic at some point.

Corollary 1.2. *If $R \subset \Omega$ is an irreducible real analytic set of constant dimension, which is complex analytic near some point $p \in R$, then R is a complex analytic subset of Ω.*

Again, Cartan's example $\{x_2(x_1^2 + y_1^2) = x_1^3\} \subset \mathbb{C}^2$ provides an irreducible real analytic set, not of constant dimension, which is complex analytic only on some part of it. Our main application concerns CR-continuation of continuous maps whose graphs are real analytic.

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Theorem 1.3. *Let M be a real analytic CR manifold. Let $f : M \to \mathbb{C}^N$ be a continuous map whose graph Γ_f is a real analytic subset of $M \times \mathbb{C}^N$, $N \geq 1$. Suppose that f is CR on a non-empty open subset of M. Then f is a CR map.*

Note that the real analytic set Γ_f is not assumed to be non-singular, and therefore the map f need not be smooth. In this case the condition for a continuous function to be CR is understood in the sense of distributions. Further, even if Γ_f is smooth, the map f may still be non-smooth, for example $f = x_2^{1/3}$ is a CR function on $M = \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 = 0\}$, its graph is a smooth real analytic set, but f is not differentiable at $\{x_2 = 0\}$.

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2. Real analytic and subanalytic sets

In this section we briefly review basic facts about real analytic sets. A real analytic set R in an open set $\Omega \subset \mathbb{R}^n$ is locally (i.e. in a neighbourhood of each point in Ω) defined as the zero locus of finitely many real analytic functions. R is called irreducible if it cannot be represented as a union of two real analytic sets each not equal to R. The germ R_p of a real analytic set R at a point $p \in R$ has a naturally defined complexification R^c_p. If $\mathbb{R}^n$ is viewed as as subset of $\mathbb{C}^n$, then $R^c_p \subset \mathbb{C}^n$ is the complex analytic germ at p characterized by the property that any holomorphic germ at p which vanishes on R_p necessarily vanishes on R^c_p. Thus R^c_p is the germ of a complex analytic set of real dimension twice that of R at p. To define complexification of real analytic sets in $\mathbb{C}^n$ it is convenient to introduce the following construction.

Let $\mathfrak{d} : \mathbb{C}^n_\zeta \to \mathbb{C}^{2n}_{(z,w)}$ be the map defined by $\mathfrak{d}(\zeta) = (\zeta, \overline{\zeta})$. Then $\mathfrak{D} := \mathfrak{d}(\mathbb{C}^n)$ is a totally real embedding of $\mathbb{C}^n$ into $\mathbb{C}^{2n}$. Suppose R is a real analytic set of dimension $d > 0$, $p \in R$, $U \subset \mathbb{C}^n$ is some neighbourhood of p, and

$$R \cap U = \left\{\varphi_j(\mathrm{Re}\zeta, \mathrm{Im}\zeta) = \varphi_j(\zeta, \overline{\zeta}) = 0, \quad j = 1, \ldots, k\right\},$$

where $\varphi_j(\zeta, \overline{\zeta})$ are real analytic in U. Then the complexification R^c_p of R_p can be defined as the germ at $\mathfrak{d}(p)$ in $\mathbb{C}^{2n}$ of the smallest complex analytic set which contains the germ of $\mathfrak{d}(R)$ at $\mathfrak{d}(p)$. When U and φ_j are suitably chosen, the complexification may simply be given by a representative

$$\{(z, w) \in U' : \varphi_j(z, w) = 0\},$$

where $U' \subset \mathbb{C}^{2n}$ is some neighbourhood of $\mathfrak{d}(p)$. Thus R^c_p is the germ of a complex analytic set of complex dimension d such that $R^c_p \cap \mathfrak{D} = \mathfrak{d}(R_p)$. If R_p is irreducible, then so is R^c_p. The following proposition allows to replace germs of analytic sets with their representatives. The proof can be found in [11].

Proposition 2.1. *Let $R \subset \Omega$ be a real analytic set. Then for every point $p \in R$ there is a neighbourhood U of p such that if Q is a real analytic set in Ω and $R_p \subset Q_p$, then $R \cap U \subset Q \cap U$.*

It follows from Proposition 2.1 that the function $d(p) = \dim X_p$ is upper semicontinuous on R. For $R \subset \Omega$ real analytic denote by $\mathcal{O}^{\mathbb{R}}(\Omega)$ the sheaf of germs of real analytic functions, and by $\mathcal{I}(R)$ the ideal in $\mathcal{O}^{\mathbb{R}}(\Omega)$ of germs of real analytic functions that vanish on R (the so-called sheaf of ideals of R). Then R is called coherent if $\mathcal{I}(R)$ is a coherent sheaf of $\mathcal{O}^{\mathbb{R}}$-modules. In fact, it follows from Oka's theorem (which also holds in the real analytic category) that R is coherent if the sheaf $\mathcal{I}(R)$ is *locally finitely generated*. The latter means that for every point $a \in R$ there exists an open neighbourhood $U \subset \Omega$ and a finite number of functions φ_j, real analytic in U and vanishing on R, such that for any point $b \in U$, the germs of φ_j at b generate the ideal $\mathcal{I}(R_b)$. Note that the corresponding statement for complex analytic sets always holds by Cartan's theorem, i.e. every complex analytic set is coherent.

Proposition 2.2. *Let $\Omega \subset \mathbb{C}^n$ be an open set, and let $R \subset \Omega$ be an irreducible real analytic set of constant positive dimension d. Suppose $R^c_a = A_{\mathfrak{d}(a)}$, where $A_{\mathfrak{d}(a)}$ is a germ at point $\mathfrak{d}(a)$ of some irreducible complex analytic set A defined in some open set in $\mathbb{C}^{2n}$. Then for any point $b \in R$ sufficiently close to a,*

(i) $R^c_b \subset A_{\mathfrak{d}(b)}$, where $A_{\mathfrak{d}(b)}$ is the germ of A at $\mathfrak{d}(b)$. Further, $\dim R^c_b = \dim A_{\mathfrak{d}(b)}$, and thus R^c_b is the union of certain irreducible components of $A_{\mathfrak{d}(b)}$.

(ii) If R is coherent, then $R^c_b = A_{\mathfrak{d}(b)}$.

Proof. (i) Let A be an irreducible complex analytic subset of some open set $U' \subset \mathbb{C}^{2n}$, $\mathfrak{d}(a) \in U'$, such that $A_{\mathfrak{d}(a)} = R^c_a$. It follows from Proposition 2.1 that if U' is sufficiently small, then $\mathfrak{d}(R) \cap U' \subset A$. In particular, if $b \in R$ is close to a, then $\mathfrak{d}(R_b) \subset A$, and therefore, $R^c_b \subset A$. Now, since A and R^c_b both have dimension d and contain $\mathfrak{d}(R_b)$, which is a totally real subset of real dimension d, it follows that R^c_b must coincide with the union of some irreducible components of $A_{\mathfrak{d}(b)}$.

(ii) Since R is coherent, there exist functions $\varphi_1(\zeta, \overline{\zeta}), \ldots, \varphi_k(\zeta, \overline{\zeta})$ real analytic in some open set $U \subset \mathbb{C}^n$ containing a, such that the germs of these functions at any point $b \in U$ generate the ideal $\mathcal{I}(R_b)$. Then R^c_b is defined by the equations $\varphi_j(z, w) = 0$. By the uniqueness theorem for complex analytic sets, $R^c_b = A_b$. $\square$

In Proposition 2.2(ii) the assumption that R is coherent cannot be in general replaced by the assumption that R has constant dimension. Indeed, consider the set (cf. [3])

$$R = \{x \in \mathbb{R}^3 : x_3(x_1^2 + x_2^2)(x_1 + x_2) = x_1^4\}. \tag{2}$$

This is an irreducible real analytic set of constant dimension which is not coherent at the origin. We may naturally identify $\mathbb{R}^3$ with the set $\mathbb{C}^3 \cap \{y_1 = y_2 = y_3 = 0\}$. Then the set

$$X = \{z = x + iy \in \mathbb{C}^3 : z_3(z_1^2 + z_2^2)(z_1 + z_2) = z_1^4\}$$

can be viewed as the complexification of R at the origin. However, at any point $p = (0, 0, x_3),\ x_3 \neq 0$, the germ X_p is reducible, and only one of its components is the complexification of R_p, which is irreducible. The set R in (2) (viewed as a subset of $\mathbb{C}^3$) is also an example of an irreducible non-coherent real analytic set which has constant dimension and such that there is no globally defined X such that X_p is the smallest complex analytic germ containing R_p for all $p \in R$.

In the proof of Theorem 1.3 we will use some results concerning subanalytic sets. A subset R of a real analytic manifold M is called *semianalytic* if for any point $p \in M$ there exist a neighbourhood U and a finite number of functions φ_{jk} and ψ_{jk} real analytic in U such that

$$R \cap U = \cup_j \{\zeta \in U : \varphi_{jk}(\zeta) = 0,\ \psi_{jk}(\zeta) > 0,\ k = 1, \ldots, l\}.$$

In particular, a real analytic set is semianalytic. A subset S of a real analytic manifold M is called *subanalytic* if for any point in M there exists a neighbourhood U such that $S \cap U$ is a projection of a relatively compact semianalytic set, that is there exists a real analytic manifold N and a relatively compact semianalytic subset R of $M \times N$ such that $S \cap U = \pi(R)$, where $\pi : M \times N \to M$ is the projection.

The main reason for introducing the class of subanalytic sets comes from the fact that the images of semianalytic (in particular real analytic) sets under real analytic maps are subanalytic. Semi- and subanalytic sets enjoy many properties of real analytic sets, for example, a finite union, intersection and set-theoretic complement of such sets is again in the same class. Further, semi- and subanalytic sets admit stratifications satisfying certain properties.

Given a subanalytic set S, we call a point $p \in S$ *regular* if near p the set S is just a real analytic manifold of dimension equal to that of S (i.e. maximal possible). Denote by $\operatorname{reg}(S)$ the set of all regular points. Points which are not regular form the set of singular points, $\operatorname{sing}(S)$. We will use the following result due to Tamm, [12], Theorem 1.2.2(v).

Proposition 2.3 ([12])**.** *If S is a subanalytic set, then $\operatorname{reg}(S)$ and $\operatorname{sing}(S)$ of S are both subanalytic. Moreover, $\dim \operatorname{sing}(S) \leq \dim S - 1$, unless $S = \emptyset$.*

For more about semianalytic and subanalytic sets see e.g. [2] or [8]. A detailed discussion of real analytic sets can be found in [3], [10] or [11].

3. Pre-images of projections of analytic sets

Let $R \subset \mathbb{C}^n$ be a real analytic set, $0 \in R$, and R irreducible at the origin. Let $A \subset \mathbb{C}^{2n}$ be a representative of R^c_0, the complexification of the germ R_0, and let $\mathfrak{d} : \mathbb{C}^n_\zeta \to C^{2n}_{(z,w)}$ be the totally real embedding. After rescaling we may assume that A is an irreducible complex analytic subset of the unit polydisc $\Delta^{2n} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : |z_j| < 1,\ |w_j| < 1,\ j = 1, \dots, n\}$, and $\mathfrak{d}(R) \cap \Delta^{2n} \subset A$. Let

$$\pi : \Delta^{2n}_{(z,w)} \to \Delta^n_z \tag{3}$$

be the coordinate projection onto z-subspace. For the proof of Theorem 1.1 we will need the following result.

Lemma 3.1. *There exist a closed nowhere dense subset S of A, a neighbourhood U of $A \setminus S$ in Δ^{2n}, and a complex analytic subset Y of U with the following properties:*

(a) S does not divide A, and $\mathfrak{d}(R) \cap S$ is nowhere dense in $\mathfrak{d}(R)$.

(b) Y can be locally given as the zero locus of a system of holomorphic equations each of which is independent of the variable w.

(c) $A \cap U \subset Y \subset \pi^{-1}(\pi(A))$.

Proof. For $1 \le j \le n$ define

$$\pi_j : \mathbb{C}^{2n}_{(z,w)} \to \mathbb{C}^{2n-1}_{(z,w_1,\dots,w_{j-1},w_{j+1},\dots,w_n)} \tag{4}$$

to be the coordinate projection parallel to w_j-direction, and observe that for any $1 \le k \le n$,

$$\pi_k^{-1} \circ \pi_k \circ \pi_{k-1}^{-1} \circ \pi_{k-1} \dots \pi_1^{-1} \circ \pi_1(A) \subset \pi^{-1}(\pi(A)). \tag{5}$$

For $p \in A$ let $l_p\pi_1$ denote the germ of the fibre of $\pi_1|_A$ at p, i.e. the germ at p of the set $\pi_1^{-1}(\pi_1(p)) \cap A$. Then by the Cartan-Remmert theorem (see e.g. [9]) the set

$$E_1 = \{p \in A : \dim l_p\pi_1 > 0\} \tag{6}$$

is complex analytic. Suppose that $\dim E_1 = \dim A$. Then, since A is irreducible, $E_1 = A$, and therefore every point of A has a fibre of dimension one. It follows then that $\pi_1^{-1}(\pi_1(A)) = A$, and so $\pi_1^{-1}(\pi_1(A))$ is a complex analytic subset of Δ^{2n}.

Assume now that $\dim E_1 < \dim A$. Then $\mathfrak{d}(R) \not\subset E_1$, since otherwise $\mathfrak{d}(R)$ would be contained in a complex analytic set of dimension smaller than that of A, but A is a representative of the complexification of R_0, which means that A is the smallest complex analytic set containing $\mathfrak{d}(R)$. Since every point in E_1 has a fibre of dimension one, $\pi_1^{-1}(\pi_1(E_1)) = E_1$, and if $p \in A \setminus E_1$, then $\pi_1(p) \notin \pi_1(E_1)$.

Suppose that $p_0 \in A \setminus E_1$. Then $\pi_1^{-1}(\pi_1(p_0)) \cap A$ is a finite set. Therefore, there exists a neighbourhood $\Omega_0 \subset \Delta^{2n}$ of p_0 such that the restriction of π_1 to $\Omega_0 \cap A$ is a finite map. Furthermore, we can choose Ω_0 to be of the form $\Omega_0 = \Omega_0' \times \Omega_0'' \subset \Delta^{2n-1} \times \Delta_{w_1}$, with the property that $A \cap (\Omega_0' \times (\partial\Omega_0'')) = \varnothing$. It follows that $\pi_1|_{A\cap\Omega_0}$ is a proper map, and by the Remmert proper mapping theorem, $\pi_1(A \cap \Omega_0)$ is a complex analytic subset of Ω_0'. Hence, $\pi_1^{-1}(\pi_1(A \cap \Omega_0))$ is a complex analytic subset of Ω_0. Furthermore,

$$\pi_1^{-1}(\pi_1(A \cap \Omega_0)) \cap \Omega_0 = ((\pi_1(A \cap \Omega_0)) \times \Delta_{w_1}) \cap \Omega_0 \tag{7}$$

defines a complex analytic set that can be given by a system of equations independent of w_1. We set $S_1 = E_1$, and thus we proved that there exists a neighbourhood U_1 of $A \setminus S_1$ and a complex analytic subset Y_1 of U_1 that locally can be represented as in (7). It follows from (5) that $Y_1 \subset \pi^{-1}(\pi(A))$.

We now argue by induction. Suppose that there exist a closed nowhere dense subset S_k of A, which satisfies condition (a) of the lemma, and a complex analytic subset Y_k of some neighbourhood U_k of $A \setminus S_k$, which satisfies (c) and can be locally given in the form $\{\varphi(z, w_{k+1}, \dots w_n) = 0\}$, $k < n$. If the set Y_k is reducible, we keep only one irreducible component of Y_k which contains $A \cap U_k$, and for simplicity denote this component again by Y_k. (Observe that $A \cap U_k$ is still irreducible, since the regular part of $A \cap U_k$ is connected.) Consider the set

$$E_{k+1} = \{p \in Y_k : \dim l_p\pi_{k+1} > 0\}. \tag{8}$$

Then as before, E_{k+1} is a complex analytic subset of Y_k. If $\dim E_{k+1} = \dim Y_k$, then $E_{k+1} = Y_k = \pi_{k+1}^{-1}(\pi_{k+1}(Y_k))$, and

$$\pi_{k+1}^{-1}(\pi_{k+1}(Y_k)) = \left(\pi_{k+1}^{-1}(\pi_{k+1}(Y_k)) \cap \{(z,w) : w_{k+1} = 0\}\right) \times \Delta_{w_{k+1}}.$$

This show that Y_k is a complex analytic subset of U_k that satisfies (c), and can be given by a system of equations independent of $(w_1, \dots, w_{k+1})$.

Suppose now that $\dim E_{k+1} < \dim Y_k$. If $A \cap U_k \subset E_{k+1}$, then we simply define $Y_{k+1} = E_{k+1}$. This defines a complex analytic subset of U_k with all the required properties. The remaining case is $A \not\subset E_{k+1}$. Then every point $a \in A \setminus E_{k+1}$ is contained in some neighbourhood in which the map $\pi_{k+1}|_{Y_k}$ is proper, and we may repeat the argument above to define Y_{k+1} to be a complex analytic subset of some neighbourhood of $A \setminus S_{k+1}$, where $S_{k+1} = S_k \cup (A \cap E_{k+1})$, such that locally Y_{k+1} is given as $\pi_{k+1}^{-1}(\pi_{k+1}(Y_k))$. By assumption, S_{k+1} is closed nowhere dense in A, does not divide A, and does not contain $\mathfrak{d}(R)$. By construction, Y_{k+1} is locally given as $\pi_{k+1}(Y_k) \times \Delta_{w_{k+1}}$, and therefore it can be defined by a system independent of $(w_1, \dots, w_{k+1})$. Finally, by (5), $Y_{k+1} \subset \pi^{-1}(\pi(A))$.

After n steps the set Y_n defined in a neighbourhood of $A \setminus S_n$ will satisfy the lemma. □

4. Proof of Theorem 1.1

For each $p \in R$ denote by $d(p)$ the dimension of the germ X_p. Since $d(p)$ is a positive integer-valued function on R, there exists a minimum, say d. Let q be a point where $d(p)$ attains its minimum, and let X_q be the germ of a complex analytic set at q with $\dim X_q = d$ which contains R_q. Furthermore, since the set of points where R is not locally irreducible is contained in the singular part of R (and hence nowhere dense in R), the point q can be chosen in such a way that R is locally irreducible at q. This implies that X_q can also be chosen to be irreducible. Let Ω be a connected open neighbourhood of q, and $X \subset \Omega$ be a particular representative of the germ X_q such that X_p is the germ of the smallest dimension containing R_p for all $p \in R \cap \Omega$, but X is either not defined at some point $p_0 \in \partial\Omega \cap R$ or X_{p_0} does not satisfy the described property. We consider two cases depending on whether R_{p_0} is irreducible or not.

Along with the complexification of R we may also consider the complexification in $\mathbb{C}^{2n}$ of the set X. If $\{\varphi_k(\zeta) = 0\}$ is the system of holomorphic equations defining X, then the system $\{\varphi_k(z) = 0\}$ (variables w_j are not involved) defines a complex analytic set X_z on some open set in $\mathbb{C}^{2n}$. We note that $X_z \cap \mathfrak{D} = \mathfrak{d}(X)$, and that the canonical complexification of X (viewed as a real analytic set) can be recovered from X_z as

$$X^c = X_z \cap X_w := \{\varphi_k(z) = 0\} \cap \{\overline{\varphi}_k(w) = 0\}. \tag{9}$$

In particular, $X^c \subset X_z$ on a non-empty open set in $\mathbb{C}^{2n}$ where X^c and X_z are both well-defined.

Conversely, if Y is a complex analytic set in $\mathbb{C}^{2n}$ defined by a system of equations $\varphi_k(z) = 0$, which are independent of w, then Y induces a complex analytic set X in $\mathbb{C}^n$ that can be defined as

$$X := \mathfrak{d}^{-1}\left(\mathfrak{D} \cap \{\varphi_k(z) = 0\} \cap \{\overline{\varphi}_k(w) = 0\}\right).$$

If fact, $\mathfrak{D} \cap \{\varphi(z) = 0\} \cap \{\overline{\varphi}(w) = 0\} = \mathfrak{D} \cap Y$, and $X = \mathfrak{d}^{-1}(\mathfrak{D} \cap Y)$.

Consider first the case when R_{p_0} is irreducible. After a translation we may assume that $p_0 = 0$, and so any neighbourhood of the origin contains an open piece of X. To simplify the exposition we denote by A some representative of R_0^c and assume without loss of generality that A is a complex analytic subset of the polydisc Δ^{2n} defined by a system of equations holomorphic in Δ^{2n}. Here $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_n) \in \mathbb{C}^n \times \mathbb{C}^n$, and the complexification of R comes from identifying ζ with z, and $\overline{\zeta}$ with w. The set A can be chosen irreducible. Let π be as in (3). By Lemma 3.1 there exist a closed nowhere dense subset $S \subset A$, which does not divide A and $\mathfrak{d}(R) \not\subset S$, a neighbourhood U of $A \setminus S$ in Δ^{2n}, and a complex analytic subset Y of U, which may be locally defined by a system of equations independent of w, such that $A \cap U \subset Y \subset \pi^{-1}(\pi(A))$. Since U is connected, and Y can be chosen irreducible, we may assume that Y has constant dimension.

Let $p \in R \cap X$ be arbitrarily close to the origin, and $\mathfrak{d}(p) \in \Delta^{2n} \setminus S$. By Proposition 2.2(i), there exists a neighbourhood $V \subset \Delta^{2n}$ of $\mathfrak{d}(p)$ such that certain components of $A \cap V$ coincide with $R_p^c \cap V$. Denote them by $\tilde{A}$. Since $R_p^c \subset X^c \subset X_z$, we conclude that $\tilde{A} \subset X_z$. Therefore, $\pi^{-1}(\pi(\tilde{A})) \subset X_z$. Indeed,

$$\pi^{-1}(\pi(\tilde{A})) \subset \pi^{-1}(\pi(X_z)) = X_z,$$

where the last equality holds because X_z is defined by a system of equation independent of w. On the other hand, X is the smallest complex analytic germ containing R, and therefore, $\pi^{-1}(\pi(\tilde{A})) = X_z$, as otherwise, the set $\pi^{-1}(\pi(\tilde{A}))$ would induce a smaller complex analytic set in $\mathbb{C}^n$ containing R_p.

We now claim that $\dim Y = \dim X_z$. First, observe that

$$\dim \pi(R_p^c) = \dim \pi(A). \tag{10}$$

Indeed, suppose that on the contrary, $\dim \pi(R_p^c) < \dim \pi(A)$. Let k be the generic dimension of the fibre $\pi^{-1}(z) \cap A$ for $z \in \pi(A)$. Then since $\dim R_p^c = \dim A$,

$$R_p^c \subset \{a \in A : \dim l_a \pi > k\}.$$

The latter is a complex analytic subset of A by the Cartan-Remmert theorem, and by the assumption, it is a proper subset of A. But this contradicts irreducibility of A. Thus (10) holds, which implies $\dim \pi^{-1}(\pi(R_p^c)) = \dim Y$. But $\dim \pi^{-1}(\pi(R_p^c)) = \dim X_z$, as otherwise, $\dim \pi^{-1}(\pi(R_p^c))$ induces a complex analytic set in $\mathbb{C}^n$ which contains R_p and which is smaller than the set induced by X_z. This proves the claim. Finally, since $\pi^{-1}(\pi(\tilde{A})) = X_z$, the set $Y \cap V$ contains $X_z \cap V$ as a union of locally irreducible components at p.

Thus we proved that if R is locally irreducible at $p_0 = 0$, then for any point $p \in R \setminus \mathfrak{d}^{-1}(\mathfrak{d}(R) \cap S)$, the set $\mathfrak{d}^{-1}(Y \cap \mathfrak{D})$ defines a complex analytic germ in $\mathbb{C}^n$ at p which has dimension d and contains R_p. To complete the proof of part (i) of Theorem 1.1 it remains to consider the case when R_{p_0} is reducible. Note that the above construction produces a complex analytic germ of dimension d which contains a dense open subset of one of the irreducible components of R_{p_0}.

We claim that given any two points on R there exists a path $\gamma \subset R$ that connects these points and satisfies the property that if $a \in \gamma$ is a point where R is locally reducible, then γ stays in the same local irreducible component of R at a. Arguing by contradiction, denote by Σ the set of all points on R that can be connected with a given point q by a path γ which satisfies the above property, and suppose that $\Sigma \neq R$. We claim that Σ is a real analytic set. Indeed, if $p \in \Sigma$ is a smooth point of R, then clearly a full neighbourhood of p in R is contained in Σ. If $p \in \Sigma$ is a singular point of R, then either R_p irreducible, in which case again a full neighbourhood of p in R is contained in Σ, or R_p is reducible, and only some components of R_p are in Σ. In any case, Σ is a real analytic subset of some neighbourhood of p. Since Σ is clearly closed, it follows that it is a real analytic set. Let $\Sigma' := \overline{R \setminus \Sigma}$, where the closure is taken in R. Then Σ' is also a real analytic set. Indeed, if $a \in R \setminus \Sigma$, then a full neighbourhood of a in R is in Σ', and if $a \in (\overline{R \setminus \Sigma}) \setminus (R \setminus \Sigma)$, then $a \in R$ is a point at which R_a is reducible, and therefore near a the set Σ' coincides with some irreducible components of R_a. Since Σ' is analytic near any of its points and closed, it follows that Σ' is a real analytic set. Thus $R = \Sigma \cup \Sigma'$, but this contradicts irreducibility of R. Hence, $\Sigma = R$, and that proves the claim.

So if in the situation above R_{p_0} is reducible, we find a path γ with the described property which connects q with the points on other components of R_{p_0} and repeat the above construction along γ sufficiently many times. This proves that there exists a dense open subset U of some neighbourhood of p_0 such that for any $p \in U$, the germ R_p is contained in some complex variety of dimension d.

For the proof of part (ii) simply observe, that by Proposition 2.2(ii), $A = X^c$ near p, and therefore, Y defines analytic continuation of the set X_z to a neighbourhood of the origin. This in its turn provides analytic continuation of the set X to a neighbourhood of a dense open subset of R. Further, the same holds if we repeat the above construction near any other point on R, and therefore there exists a complex analytic set X in a neighbourhood of a dense open subset of R with the desired properties. Theorem 1.1 is proved.

Proof of Corollary 1.2. It follows from Theorem 1.1 that there exists an open set $\Omega \subset \mathbb{C}^n$ such that $R \cap \Omega$ is a complex analytic set which is dense in R. We now use the result of Diederich and Fornæss [6]. The claim in the proof of Theorem 4 in [6] states that if X_q is a complex analytic germ at $q \in R$, and $X_q \subset R$, then there exists a neighbourhood U of q, independent of X_q, such that X_q extends to a closed complex analytic subset of U. It follows that Ω can be chosen to be a neighbourhood of R, which proves that R is a complex analytic set. $\square$

5. Proof of Theorem 1.3

First note that Γ_f is an irreducible real analytic set of constant dimension. Let $\tilde{M}$ be the subset of M on which f is CR. Since the set of singular points of the real analytic set Γ_f is nowhere dense in Γ_f, there exists a point $p \in \tilde{M}$ such that $(p, f(p))$ is a smooth point of Γ_f. Further, the point p can be chosen in such a way that Γ_f near $(p, f(p))$ is the graph of a smooth map on M. In fact, by the real analytic version of the implicit function theorem (see e.g. [4]) there exists a neighbourhood U_p of p such that the map $f : M \cap U_p \to \mathbb{C}^N$ is real-analytic.

If the CR codimension of M is zero, then M is simply a complex manifold, and the map f is holomorphic in U_p. Therefore, Γ_f is complex analytic over U_p, and by Corollary 1.2 is a complex analytic set in $M \times \mathbb{C}^N$. Since the projection $\pi : \Gamma_f \to M$ is injective, it follows (see e.g. [5]) that Γ_f is itself a complex manifold, $\pi : \Gamma_f \to M$ is biholomorphic, and therefore, $f = \pi' \circ \pi^{-1}$ is holomorphic everywhere on M (here $\pi' : \Gamma_f \to \mathbb{C}^N$ is another projection). If the CR dimension of M is zero, then there is nothing to prove since any function is CR. Hence, we may assume that both the CR dimension and the CR codimension of M are positive.

The problem is local, therefore, it is enough to prove that f is CR in a neighbourhood of a point $q \in M$ which is a boundary point of $\tilde{M}$, and then use a continuation argument. By [1], there exists a neighbourhood U of q in M such that U can be generically embedded into $\mathbb{C}^n$ for some $n > 1$, i.e. n is the sum of the CR-dimension and codimension of M. Thus, without loss of generality we may assume that M is a generic real analytic submanifold of some domain in $\mathbb{C}^n$, and $f : M \to \mathbb{C}^N$ is a continuous map which is a real analytic CR map on some non-empty subset $\tilde{U}$ of M. By [13], every component f_j of $f|_{\tilde{U}}$ extends to a function F_j holomorphic in some neighbourhood of $\tilde{U}$. Then the map $F = (F_1, \dots, F_N)$ defines a complex analytic set in $\mathbb{C}^n \times \mathbb{C}^N$ of dimension n, namely, its graph Γ_F. By construction Γ_F contains the set Γ_f.

Observe that Γ_F is the smallest complex analytic set which contains Γ_f. Indeed, suppose, on the contrary, that there exists a complex analytic set A, $\dim A < n$, which contains a non-empty subset of Γ_f. Let $\pi : \mathbb{C}^{n+N} \to \mathbb{C}^n$ be the projection. Then $M \subset \pi(A)$, where $\pi(A)$ is a countable union of complex analytic sets in $\mathbb{C}^n$ of dimension at most $n-1$. This is however impossible, since M is a generic submanifold of $\mathbb{C}^n$.

We now show that f is CR everywhere on M. By Theorem 1.1, there exist a closed nowhere dense set $S \subset \Gamma_f$, and a complex analytic subset X of a neighbourhood of $\Gamma_f \setminus S$, $\dim X = n$, which contains $\Gamma_f \setminus S$. Suppose first that $p \in M$, and $(p, p') \in \Gamma_f \setminus S \subset \mathbb{C}^n \times \mathbb{C}^N$. Choose neighbourhoods Ω and Ω' of p and p' respectively, so that $X \cap (\Omega \times \Omega')$ is complex analytic. Let $\pi : X \to \Omega$ be the projection, and let

$$E = \{x \in X : \dim l_x \pi > 0\}.$$

Let k be the generic dimension of $\pi^{-1}(\zeta) \cap X$ for $\zeta \in \pi(X)$. Then $k = 0$, since otherwise, $\pi(X)$ is a locally countable union of complex analytic sets of dimension at most $n - k < n$, and therefore $\pi(X)$ cannot contain a generic submanifold M. Therefore, $\dim E < n$, and in particular, $M \not\subset \pi(E)$. Consider the set $E \cap \Gamma_f$. This is a real analytic subset of $\Omega \times \Omega'$, and therefore, its projection, $T := \pi(E \cap \Gamma_f)$, is a subanalytic set in Ω. From the above considerations, $T \neq M$.

We first show that for any point $\zeta \in (M \cap \Omega) \setminus T$, the map f is CR at ζ. Since $\pi(E)$ is closed, there exists a neighbourhood $V \subset \Omega$ of ζ such that $V \cap \pi(E) = \varnothing$. We may further shrink V and choose a neighbourhood V' of $f(\zeta)$ such that the projection $\pi : X \cap (V \times V') \to V$ is proper. In particular this implies that π is a branched covering. Let $G \subset X \cap (V \times V')$ be the branch locus of π. Then near any point $a \in (M \cap V) \setminus \pi(G)$ the map $\pi^{-1} : V \to X$ splits into a finite number of holomorphic maps. It follows that there exists a branch of π^{-1}, say σ, such that near a, the map f coincides with the restriction to M of a holomorphic map $\pi' \circ \sigma : V \to \mathbb{C}^N$. Therefore f is CR. The set $M \cap \pi(G)$ is the intersection of a real analytic manifold and a complex analytic subset of V, and therefore it admits a stratification into a finite number of smooth components. Each of these components is a removable CR-singularity for f. More precisely one has the following result.

Lemma 5.1. *Let M be a smooth generic submanifold of $\mathbb{C}^n$, of positive CR dimension and codimension. Let $S \subset M$ be a smooth submanifold with $\dim S < \dim M$. Then any function h continuous on M and CR on $M \setminus S$ is CR on M.*

This is a trivial generalization of Proposition 4 in [7] where the result is stated for real hypersurfaces. The proof is the same, with the only difference being the degree of the form φ. Applying Lemma 5.1 to each smooth component of $M \cap \pi(G)$ we deduce that f is CR on $M \setminus T$.

To prove that T is a removable CR-singularity for f we observe that by Proposition 2.3, the regular part of T, is a smooth submanifold of M, and therefore, by Lemma 5.1, $\operatorname{reg}(T)$ is a removable singularity for the map f. The set $\operatorname{sing}(T)$ is subanalytic of dimension strictly less than that of T, and we may repeat the process by induction. After finitely many steps, we conclude that f is CR near p.

To complete the proof of the theorem it remains to consider the case $(p,p') \in \Gamma_f \cap S$. We recall the construction in Lemma 3.1. Let A be some representative of the complexification of the germ $(\Gamma_f)_{(p,p')} \subset \mathbb{C}^{2n+2N}$. In the notation of Lemma 3.1, let k be the smallest integer such that (i) Y_k defines a complex analytic subset in a neighbourhood of $A \setminus \tilde{S}$, where $\tilde{S} = S_1 \cup \cdots \cup S_k$, and S_j are defined as in the proof of Lemma 3.1, (ii) $A \subset Y_k \subset \pi^{-1}_{(w,w')}(\pi_{(w,w')}(A))$, where $\pi_{(w,w')} : \mathbb{C}^{2n+2N}_{(z,z',w,w')} \to \mathbb{C}^{n+N}_{(z,z')}$, and (iii) Y_k is locally defined by a system of equations independent of (w,w'). Then for $\mathfrak{d} : \mathbb{C}^{n+N} \to \mathbb{C}^{2n+2N}$, $\mathfrak{d}(S) \subset \tilde{S}$ near $\mathfrak{d}(p,p')$, and for any point $q \in A \setminus \tilde{S}$, there exists a small neighbourhood of q where $Y_k = \{\varphi_\nu(z,z') = 0\}$, for some $\varphi_\nu(z)$ holomorphic near q. Let

$$a \in S_k \setminus (\bigcup_{j=1}^{k-1} S_j) \cap \mathfrak{d}(\Gamma_f). \tag{11}$$

Then there exists a neighbourhood U of $\mathfrak{d}^{-1}(a)$ such that near any point $b \in \mathfrak{d}^{-1}((A \setminus S_k) \cap \mathfrak{D}) \cap U \subset \Gamma_f$, the set Γ_f is contained in a complex analytic set of dimension n, and $\mathfrak{d}^{-1}(S_k \cap \mathfrak{d}(\Gamma_f))$ is a real analytic subset of $\Gamma_f \cap U$. We now may repeat the argument which we used to prove that T is a removable CR-singularity for f. Indeed, it follows that f is CR in $\pi(((\Gamma_f \setminus \mathfrak{d}^{-1}(S_k))) \cap U)$, and $\pi(\Gamma_f \cap \mathfrak{d}^{-1}(S_k))$ is a subanalytic set in M. Using Proposition 2.3 and Lemma 5.1 we show that f is CR on every smooth component of $\pi(\Gamma_f \cap \mathfrak{d}^{-1}(S_k))$.

This shows that Γ_f is the graph of a CR map at every point of $\mathfrak{d}^{-1}(S_k) \cap \Gamma_f$. By construction, the set S_{k-1} is complex analytic, and therefore, $\mathfrak{d}^{-1}(S_{k-1}) \cap \Gamma_f$ is real analytic, and therefore the same procedure as before applies. Arguing by induction we show that for all $j = 1, ..., k$ the set $\pi(\Gamma_f \cap \mathfrak{d}^{-1}(S_j))$ is a removable CR-singularity for f. This completes the proof of Theorem 1.3.

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