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Uniformization of strictly pseudoconvex domains. II

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Abstract. It is shown that if two strictly pseudoconvex Stein domains with real-analytic boundaries have biholomorphic universal coverings, then their boundaries are locally biholomorphically equivalent.

§ 1. Introduction

The "if" part of the following theorem was established in our paper [10]. The purpose of the present paper is to give a completely independent proof of the "only if" part.

Main Theorem. *Let D and D' be strictly pseudoconvex Stein domains with real-analytic boundaries. Then the universal coverings of D and D' are biholomorphic if and only if the boundaries of these domains are locally biholomorphically equivalent.*

As in [10], two situations should be considered separately. In the generic "non-spherical" case we have a stronger result that may be viewed as a generalization of the boundary correspondence theorem.

Theorem 1.1. *If the universal coverings of open domains D and D' are not biholomorphic to the unit ball, then any biholomorphism between them extends to a biholomorphism of the universal coverings of the closed domains $\overline{D}$ and $\overline{D'}$.*

This statement is clearly false for domains non-trivially covered by the unit ball $B \subset \mathbb{C}^n$. Indeed, let $D = D'$ be one of the domains constructed in [1] and [4]. The closure of D is universally covered by a region of the form $\overline{B} \setminus A$, where A is a non-empty closed subset of the unit sphere (see also [10], § 4.2). Taking any biholomorphism of the unit ball whose extension to ∂B does not preserve A, we get an example of a biholomorphism of the universal covering of D which does not extend to a biholomorphism of the universal covering of $\overline{D}$. However, the situation does not get worse than this because of the following converse to Theorem A.2 of [10].

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Theorem 1.2. *If a strictly pseudoconvex Stein domain with real-analytic boundary is covered by the unit ball, then its boundary is spherical, that is, it is locally biholomorphically equivalent to the unit sphere.*

The proofs of both of these theorems make essential use of ideas and methods developed by Pinchuk in his paper [12].

Theorems 1.1 and 1.2 are proved in § 2. An extension of the Wong–Rosay theorem to coverings is obtained in the process. In § 3 we discuss similar results for domains with smooth boundaries and for strictly pseudoconvex non-Stein domains.

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§ 2. Coverings and their biholomorphisms

2.1. Proof of Theorem 1.2. Let D be a strictly pseudoconvex Stein domain with real-analytic boundary, and let $\pi\colon B\to D$ be a covering of D by the unit ball $B\subset\mathbb{C}^n$. We equip $\overline{B}$ and $\overline{D}$ with Riemannian metrics, smooth up to the boundary, and denote the distance with respect to these metrics by $\operatorname{dist}(\,\cdot\,,\cdot\,)$.

Let $q\in\partial D$ be an arbitrary point and let V be a coordinate neighbourhood of q such that $V\cap D$ is simply connected. Then the germ of the map π^{-1} extends to a biholomorphic map g from $V\cap D$ to an open subset of B.

Let $\varphi\colon\overline{D}\to(-\infty,0]$ be a smooth plurisubharmonic defining function for D. Applying the Hopf lemma to the negative plurisubharmonic function $\varphi\circ\pi$ in the ball, we see that

$$c_1\operatorname{dist}(g(x),\partial B)\leqslant|\varphi\circ\pi(g(x))|=|\varphi(x)|\leqslant c_2\operatorname{dist}(x,\partial D)$$

for any point $x\in V\cap D$ with positive constants c_1 and c_2 that do not depend on x (see [12], § 1.1.1 and § 2.3 below).

Using a standard argument based on the asymptotic behaviour of the Kobayashi metric on B (see [12], § 2.1.2 or [11], [3], [15]), we see that g extends to $\partial D\cap V$ as a Hölder continuous map sending $\partial D\cap V$ to the unit sphere. We note that this extension to the boundary is non-constant by the boundary uniqueness theorem. Hence the extension of g to ∂D is in fact smooth by the main result of [13]. Finally, the reflection principle of Lewy and Pinchuk yields that the map g extends biholomorphically to a neighbourhood of q. Hence ∂D is locally biholomorphically equivalent to the sphere at every boundary point $q\in\partial D$.

Remark 2.1. Since the argument works for *any* local inverse g of π, it follows directly (that is, without any appeal to [10], § 4.2) that the closure of D is universally covered by a region of the form $\overline{B}\setminus A$. Here A is the closed subset of the unit sphere ∂B defined as the complement in ∂B to the union of the images of all possible extensions of π^{-1} to the boundary of D.

2.2. The Wong–Rosay theorem for coverings. Let D be a strictly pseudoconvex Stein domain with C^2-smooth boundary. Then D is complete hyperbolic in the sense of Kobayashi by (a slight extension of) a result of Graham [5] (see also [8]). If $\pi\colon U\to D$ is an unramified covering, then the complex manifold U is also complete hyperbolic in the sense of Kobayashi by a theorem of Eastwood [2]. In particular, Montel's theorem for holomorphic maps to complete hyperbolic spaces [7] tells us that a sequence of holomorphic maps $f_\nu\colon N \to U$ from a connected complex manifold N is relatively compact if and only if the sequence of points $f_\nu(z) \in U$ is relatively compact for some point $z \in N$.

The following proposition is a version of a result of Wong [16] and Rosay [14] for coverings of strictly pseudoconvex domains.

Proposition 2.2. *Let $\pi\colon U \to D$ be a covering of a strictly pseudoconvex Stein domain with C^2-smooth boundary. Assume that there is a point $z_0 \in U$ and a sequence of biholomorphic maps $g_\nu \in \operatorname{Aut}(U)$ such that $\pi(g_\nu(z_0)) \to \zeta_0 \in \partial D$ as $\nu \to \infty$. Then U is biholomorphic to the ball.*

Proof. Pinchuk's scaling method can be applied to the sequence of maps $f_\nu := \pi \circ g_\nu\colon U \to D$ in much the same way as in the proof of Theorem 9 in [12]. The only departure concerns the definition and convergence of the sequence of inverse maps used at the end of the proof.

Namely, if $V \ni \zeta_0$ is a coordinate neighbourhood such that $V \cap D$ is convex (and therefore simply connected), then there is a uniquely defined sequence of inverse maps $(f_\nu)^{-1}\colon V \cap D \to U$ such that $(f_\nu)^{-1}(f_\nu(z_0)) = z_0$. Composing these maps with re-scaling maps gives a sequence of maps from (increasing subsets of) the unit ball to U taking a fixed point of the ball to z_0. This sequence is relatively compact by the theorem of Montel mentioned above.

Let us draw two corollaries from Proposition 2.2. The first will be used in the proof of Theorem 1.1.

Corollary 2.3. *Let $\pi\colon U \to D$ be a covering of a strictly pseudoconvex Stein domain with C^2-smooth boundary. Assume that U is not biholomorphic to the ball. Then for every sequence of biholomorphic maps $g_\nu \in \operatorname{Aut}(U)$ and every compact set $K \Subset U$, the projections $\pi(g_\nu(K)) \subset D$ lie in a compact subset of D that does not depend on ν.*

Proof. Pick a point $z_0 \in K$ and let $d < \infty$ be the diameter of K with respect to the Kobayashi metric on U. By Proposition 2.2, all the points $\pi(g_\nu(z_0))$ lie in some compact subset of D. The image of every point $z \in K$ under any of the maps $\pi \circ g_\nu\colon U \to D$ lies within Kobayashi distance d of this compact subset. This proves the corollary since D is complete with respect to the Kobayashi metric.

We recall that a covering $p\colon X \to Y$ is said to be *regular* (or *Galois*) if Y is the quotient of X by the action of the group of deck transformations of the covering. An equivalent condition is that the image of the homomorphism $p_*\colon \pi_1(X) \to \pi_1(Y)$ of fundamental groups is a normal subgroup of $\pi_1(Y)$. In particular, the universal covering is regular because $\operatorname{Im} p_* = \{e\}$.

Corollary 2.4. *Let $\pi\colon U \to D$ be a regular covering of a strictly pseudoconvex Stein domain with C^2-smooth boundary. Assume that U is not biholomorphic to the ball. Then the group Γ of deck transformations of this covering is a cocompact lattice in the Lie group $\operatorname{Aut}(U)$.*

Proof. The group $\operatorname{Aut}(U)$ with the topology of uniform convergence on compact subsets is a Lie group because U is hyperbolic in the sense of Kobayashi. It remains to prove that the quotient $\Gamma \setminus \operatorname{Aut}(U)$ is compact or, in other words, that for every sequence $f_\nu \in \mathrm{Aut}(U)$ there is a sequence $\gamma_\nu \in \Gamma$ of deck transformations such that the sequence $\gamma_\nu \circ f_\nu$ is relatively compact.

Let $\Phi \subset U$ be a fundamental domain for the action of Γ on U. Pick a point $z_0 \in U$. For any sequence $f_\nu \in \mathrm{Aut}(U)$ there is a sequence $\gamma_\nu \in \Gamma$ of deck transformations such that $\gamma_\nu(f_\nu(z_0)) \in \Phi$. The sequence of points $\pi(\gamma_\nu(f_\nu(z_0))) \in D$ is relatively compact by Proposition 2.2. However, the restriction of the projection $\pi|_{\overline{\Phi}} \colon \overline{\Phi} \to D$ to the closure of the fundamental domain in U is proper and, therefore, the sequence $\gamma_\nu(f_\nu(z_0))$ is relatively compact in U. Hence the sequence of maps $\gamma_\nu \circ f_\nu \in \mathrm{Aut}(U)$ is relatively compact by Montel's theorem.

Example 2.5. In general, neither the group $\mathrm{Aut}(U)$ nor its connected component of the identity is compact. For instance, the universal covering of the (non-spherical) strictly pseudoconvex domain

$$D = \left\{ (z,w) \in \mathbb{C}^2 \;\middle|\; (|z|-1)^2 + |w|^2 < \frac{1}{4} \right\}$$

admits an obvious effective $\mathbb{R}$-action induced by the rotation $z \mapsto e^{it}z,\ \ t \in \mathbb{R}$.

2.3. Proof of Theorem 1.1. Let D and D' be strictly pseudoconvex Stein domains with real-analytic boundaries. We denote by $\overline{\pi}\colon \overline{Y} \to \overline{D}$ the universal covering of the closure of D. Then $\overline{Y}$ is a complex manifold with (possibly non-compact) real-analytic boundary $\partial Y = \overline{\pi}^{-1}(\partial D)$. Since D is homotopy equivalent to its closure, the interior $Y = \overline{Y} \setminus \partial Y$ with the projection $\pi = \overline{\pi}|_Y$ is the universal covering of the open domain D. A similar notation will be used for the domain D'.

Suppose that $F\colon Y \to Y'$ is a biholomorphic map. To prove that any such F extends to a biholomorphic map $\overline{F}\colon \overline{Y} \to \overline{Y'}$, it suffices to show that the map $\pi' \circ F\colon Y \to D'$ extends to a locally biholomorphic map $\overline{\pi' \circ F}\colon \overline{Y} \to \overline{D'}$. Indeed, by the monodromy theorem, this extension can be lifted to a locally biholomorphic map $\overline{F}\colon \overline{Y} \to \overline{Y'}$ coinciding with F on Y. Applying this result to the inverse map $F^{-1}\colon Y' \to Y$, we conclude that $\overline{F}$ is one-to-one.

We equip $\overline{D}$ and $\overline{D'}$ with Riemannian metrics, smooth up to the boundary, and lift these metrics to $\overline{Y}$ and $\overline{Y'}$ respectively. The distance with respect to any of these metrics will be denoted by $\mathrm{dist}(\,\cdot\,,\cdot\,)$. (There is no danger of confusion.)

Let $\varphi\colon \overline{D} \to (-\infty, 0]$ be a smooth plurisubharmonic defining function for D. We define a real-valued function ψ_0 in D' by setting

$$\psi_0(z) = \sup_{\{y \in Y' \mid \pi'(y) = z\}} \varphi \circ \pi \circ F^{-1}(y).$$

Note that ψ_0 is locally the supremum of a family of negative plurisubharmonic functions. Hence its upper semicontinuous regularization

$$\psi(z) = \psi_0^*(z) = \limsup_{\zeta \to z} \psi_0(\zeta)$$

is a non-positive plurisubharmonic function on D' (see [9], § 2.9).

The crucial point is that if Y is not biholomorphic to the unit ball, then ψ is in fact *negative* everywhere in D'. Indeed, take a point $z \in D'$ and a small closed ball Q centred at z. The pre-image of this ball in Y' is the orbit of a ball $\widetilde{Q} \subset Y'$ under the action of the countable group $\Gamma' = \pi_1(D')$ of deck transformations of the universal covering $\pi'\colon Y' \to D'$. Hence the set $F^{-1}(\pi'^{-1}(Q)) \subset Y$ is the orbit of the compact set $K = F^{-1}(\widetilde{Q})$ under the action of the countable subgroup $F^{-1}\Gamma' F \subset \operatorname{Aut}(Y)$. Corollary 2.3 shows that the projection of this orbit to D (that is, the set $\{x \in D \mid x = \pi \circ F^{-1}(y), \pi'(y) \in Q\}$) is contained in a compact subset $E \Subset D$. It follows from the definitions of ψ and ψ_0 that

$$\psi(z) \leqslant \sup_{\zeta \in Q} \psi_0(\zeta) \leqslant \max_{x \in E} \varphi(x) < 0.$$

We can now apply Hopf's lemma to the negative plurisubharmonic function ψ on D'. Namely, for any point $x \in Y$ we have

$$-c_1 \operatorname{dist}(\pi' \circ F(x), \partial D') \geqslant \psi(\pi' \circ F(x))$$

for a positive constant c_1. On the other hand, we have

$$\psi(\pi' \circ F(x)) \geqslant \psi_0(\pi' \circ F(x)) \geqslant \varphi \circ \pi \circ F^{-1}(F(x)) = \varphi(\pi(x))$$

by the definition of ψ. Since φ is a smooth defining function for D, there is a positive constant c_2 such that

$$\varphi(\pi(x)) \geqslant -c_2 \operatorname{dist}(\pi(x), \partial D).$$

Finally, if $x \in Y$ is sufficiently close to ∂Y, then

$$\operatorname{dist}(\pi(x), \partial D) = \operatorname{dist}(x, \partial Y).$$

Therefore we have proved the inequality

$$\operatorname{dist}(\pi' \circ F(x), \partial D') \leqslant c_3 \operatorname{dist}(x, \partial Y) \tag{$*$}$$

with some positive constant c_3 that does not depend on $x \in Y$ provided that x is close to the boundary.

In view of the inequality $(*)$, the existence of a locally biholomorphic extension of $\pi' \circ F\colon Y \to D'$ across any boundary point $q \in \partial Y$ is established by the argument given in the last paragraph of the proof of Theorem 1.2. As explained above, this completes the proof of Theorem 1.1.

Remark 2.6. If D and D' are covered by the unit ball, then the function ψ (constructed in the proof of Theorem 1.1) can vanish identically.

§ 3. Further results

3.1. Domains with smooth boundaries. The proofs in § 2 (except for the application of the reflection principle at the end) work for strictly pseudoconvex Stein domains with smooth boundaries. Therefore we have the following results.

Corollary 3.1. *Let D and D' be strictly pseudoconvex Stein domains.*

1. *If the universal covering of D is not biholomorphic to the ball in $\mathbb{C}^n$, then any biholomorphism between the universal coverings of these domains extends to a diffeomorphism of the universal coverings of their closures.*

2. *Any biholomorphism of the universal covering of D onto the unit ball $B \subset \mathbb{C}^n$ extends to a diffeomorphism of the universal covering of $\overline{D}$ onto a region of the form $\overline{B} \setminus A$, where A is a closed subset of the unit sphere ∂B.*

The extensions provided by this corollary are, of course, CR-maps of the boundaries. If the boundaries of the domains are C^k-smooth ($k \geqslant 2$), then the extensions can be shown to be $C^{k-1/2-0}$-smooth up to the boundary (see [6]).

Conversely, the methods of [10] can easily be adapted to prove the following result.

Corollary 3.2. *Let D and D' be strictly pseudoconvex Stein domains.*

1. *Any* CR-*diffeomorphism between the universal coverings of their boundaries extends* (*in the sense explained in* [10], § 2) *to a biholomorphism of their universal coverings.*

2. *If the boundary of D is everywhere locally* CR-*diffeomorphic to the unit sphere, then any such local* CR-*diffeomorphism extends to a biholomorphism of the universal covering of D onto the unit ball.*

We note that there is no need to exclude domains with spherical boundaries from assertion 1 of Corollary 3.2 because the CR-equivalence of the boundaries is already assumed to be global.

3.2. Non-Stein domains. An inspection of the proofs of Theorems 1.1 and 1.2 shows that the assumption that the strictly pseudoconvex domains D and D' are Stein is only used in the proofs of Proposition 2.2 and Corollary 2.3 and then only to conclude that the domain and its covering are complete hyperbolic in the sense of Kobayashi. Here we explain how to get around this point in the argument.

Let X be an arbitrary complex manifold. We denote by $k_X(\,\cdot\,,\cdot\,)$ the Kobayashi pseudometric on X and put

$$N_X = \{x \in X \mid k_X(x,y) = 0 \text{ for some } y \neq x\}.$$

Clearly, we have $N_X = \varnothing$ if and only if X is hyperbolic in the sense of Kobayashi.

Lemma 3.3. *If $f\colon X \to Z$ is a locally biholomorphic map, then $f(N_X) \subset N_Z$.*

Proof. Let $W \ni x$ be a relatively compact open neighbourhood of a point $x \in X$. It follows from the definition of k_X that

$$\min_{y\in\partial W} k_X(x,y) \leqslant \inf_{y\in X\setminus W} k_X(x,y).$$

The minimum on the left-hand side is well defined because k_X is continuous and ∂W is compact. Consequently, if $x \in N_X$, then every neighbourhood of x contains a point $y \neq x$ with $k_X(x,y) = 0$.

Now let V be a neighbourhood of a point $x \in N_X$ such that the map f is injective on V. Pick a point $y \in V$ such that $y \neq x$ and $k_X(x,y) = 0$. Then $f(y) \neq f(x)$ and $k_Z(f(x), f(y)) \leqslant k_X(x,y) = 0$. It follows that $f(x) \in N_Z$.

A non-strictly pseudoconvex Stein domain D may indeed fail to be hyperbolic in the sense of Kobayashi. (The simplest example is obtained by blowing up a point in the ball.) Nevertheless, the estimates for the Kobayashi pseudometric near the boundary are still valid and, therefore, D is complete hyperbolic in the sense of Kobayashi modulo a compact subset. In other words,

1) the set N_D is compact,

2) closed balls with respect to k_D are compact.

Property 2) shows that Corollary 2.3 follows from Proposition 2.2 in the non-Stein case as well. The proof of Proposition 2.2 must be augmented by the following lemma.

Lemma 3.4. *Suppose that a covering* $\pi\colon U \to D$ *of a* (*possibly non-Stein*) *strictly pseudoconvex domain satisfies the hypotheses of Proposition* 2.2. *Then* U *is complete hyperbolic in the sense of Kobayashi.*

Proof. Let $g_\nu \in \operatorname{Aut}(U)$ be a sequence of biholomorphisms such that $\pi(g_\nu(z_0)) \to \zeta_0 \in \partial D$. The first (easy) step in the application of the scaling method of [12] shows that there is a subsequence (again denoted by g_ν) such that the maps $\pi \circ g_\nu$ converge to the constant map to ζ_0 uniformly on compact subsets of U. However, if $x \in N_U$, then $\pi(g_\nu(x)) \in N_D \Subset D$ for all ν by the previous lemma because the maps $\pi \circ g_\nu$ are locally biholomorphic. Hence the points $\pi(g_\nu(x))$ cannot converge to the boundary of D.

This contradiction shows that N_U is empty, whence U is hyperbolic in the sense of Kobayashi. Thus D is covered by a Kobayashi hyperbolic manifold and, therefore, is itself hyperbolic. Since D is already known to be complete, it follows that U is complete by Eastwood's theorem [2].

Corollary 3.5. *Theorems* 1.1 *and* 1.2 *remain true for strictly pseudoconvex domains that are not necessarily Stein.*

On the other hand, the results of [10] cannot be extended as they stand to non-Stein domains. For instance, blowing up a point in the ball gives an example of a strictly pseudoconvex domain with spherical boundary that is not universally covered by the ball. More interesting examples can be obtained by taking ramified coverings of the non-Stein quotients of the unit ball in $\mathbb{C}^2$ constructed in [4].

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