Izvestiya: Mathematics **69**:6 1189–1202
Izvestiya RAN: Ser. Mat. **69**:6 115–130 DOI 10.1070/IM2005v069n06ABEH002295

Uniformization of strictly pseudoconvex domains. I

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Abstract. It is shown that two strictly pseudoconvex Stein domains with real-analytic boundaries have biholomorphic universal coverings provided that their boundaries are locally biholomorphically equivalent. This statement can be regarded as a higher-dimensional analogue of the uniformization theorem.

§ 1. Introduction

Showing the biholomorphic equivalence of domains in complex spaces of dimension greater than one has proved to be a difficult problem. Poincaré observed that the unit ball B in $\mathbb{C}^2$ is not biholomorphically equivalent to the bidisc Δ^2. In fact, the situation may seem rather hopeless as it has been shown that almost any two randomly chosen domains in $\mathbb{C}^n$, $n > 1$, are inequivalent. On the positive side, Pinchuk discovered that the biholomorphic equivalence of domains with real-analytic boundaries is closely connected to the *local* biholomorphic equivalence of their boundaries. The purpose of this paper and its sequel [17] is to prove a fairly general result in this direction.

Main Theorem. *Let D and D' be strictly pseudoconvex Stein domains with real-analytic boundaries. Then the universal coverings of D and D' are biholomorphic if and only if the boundaries of these domains are locally biholomorphically equivalent.*

In this paper we prove the "if" part of this theorem. To do this, we establish the following extension theorem by combining the work of Pinchuk [18]–[20] and Vitushkin et al. [25], [26] with the classical theory of envelopes of holomorphy and, more specifically, with the results of Kerner [16] on the envelopes of holomorphy of coverings.

Theorem A. *Let D and D' be strictly pseudoconvex Stein domains with real-analytic boundaries. Then any local equivalence between their boundaries extends to a biholomorphism of the universal coverings of the* (*open*) *domains D and D'.*

This theorem can be regarded as two conceptually different assertions. In the generic case the boundaries are non-spherical, that is, they are nowhere locally equivalent to the unit sphere in $\mathbb{C}^n$, and a somewhat stronger result can be obtained.

The first author was partially supported by grants from the Russian Foundation for Basic Research, INTAS and the programme "Theoretical mathematics" of the Russian Academy of Sciences.

AMS 2000 *Mathematics Subject Classification.* 32D15, 32D10, 32E10.

Theorem A.1.[1] *If the domains D and D' in Theorem* A *have non-spherical boundaries, then any local equivalence between ∂D and $\partial D'$ extends to a biholomorphism from the universal covering of $\overline{D}$ to the universal covering of $\overline{D'}$.*

If the boundaries are spherical, that is, somewhere (and hence everywhere, by Pinchuk's theorem [18]) locally equivalent to the sphere, then Theorem A is essentially equivalent to the following higher-dimensional analogue of the uniformization theorem.

Theorem A.2. *A strictly pseudoconvex Stein domain with spherical boundary is universally covered by the unit ball.*

In particular, *a simply connected Stein domain with compact spherical boundary is biholomorphic to the unit ball.* This "Riemann mapping theorem" was proved by Pinchuk [18] under the stronger assumption that the *boundary* of the domain is simply connected. Twenty years later, Chern and Ji [5] established the full result for domains contained in $\mathbb{C}^n$. Quite recently, Falbel [8] observed that Pinchuk's assumption is in fact equivalent to the simple connectivity of the domain if the complex dimension $n = \dim_{\mathbb{C}} D \geqslant 3$. (Indeed, by Lefschetz' theorem for Stein manifolds [2], [3], the fundamental group of a smoothly bounded Stein domain of complex dimension $n \geqslant 3$ is isomorphic to the fundamental group of its boundary; compare §§ 2.3 and 4.2.) We note that the question remained open for general simply connected Stein domains of complex dimension two.

In fact, all the statements above have been proved for domains with simply connected boundaries by Pinchuk and others (see § 3). In this case, the analytic continuation of a local equivalence along the boundary is single-valued by the monodromy theorem, and the rest is accomplished by the standard Hartogs theorem. Our main observation is that Kerner's theorem enables us to extend multiple-valued maps directly in a Hartogs-like fashion. Then one can use the simple connectivity of (the universal covering of) the domain and obtain holomorphic maps with the desired properties.

Another application of this approach yields a generalization of a result of Ivashkovich [13] on the extension of locally biholomorphic maps from real hypersurfaces with non-degenerate indefinite Levi form in $\mathbb{CP}^n$. Here the role of the simply connected domain is played by the complex projective space itself.

Theorem B. *Let M and M' be compact real hypersurfaces with non-degenerate indefinite Levi form in $\mathbb{CP}^n$. Suppose that M is real-analytic and M' is real-algebraic. If M and M' are locally equivalent, then the equivalence map is the restriction of an automorphism of $\mathbb{CP}^n$.*

Once again, the case of a simply connected M has been treated previously [12]. The methods of [12] can be used to generalize Theorem B to generic pseudoconcave CR-submanifolds of higher codimension in $\mathbb{CP}^n$ with locally injective Segre map.

In this paper, § 2 covers the general theory of Riemann domains over complex manifolds and their envelopes of holomorphy. Kerner's theorem and some typical applications to analytic continuation are discussed in § 2.3. Then we extend

[1] *Added in proof.* F. Forstnerich has informed us that Theorem A.1 was obtained by him in 1985 in his Ph. D. thesis "Proper holomorphic mappings in several complex variables", Univ. of Washington, Seattle 1985.

Kerner's theorem to domains over $\mathbb{CP}^n$ (see § 2.4) and prove a version of results of Kerner and Ivashkovich on the extension of locally biholomorphic maps (see § 2.5). § 3 is essentially an overview of known facts about the analytic continuation of germs of biholomorphic maps between real-analytic hypersurfaces. Finally, § 4 contains proofs of the theorems stated in the introduction along with further corollaries and comments.

The authors would like to thank Sergey Ivashkovich and Seva Shevchishin for helpful discussions.

§ 2. Generalities on analytic continuation

2.1. Riemann domains over complex manifolds. A *domain over a complex manifold* X is a pair (D, p_D) consisting of a connected Hausdorff topological space D and a locally homeomorphic map $p_D \colon D \to X$. There is a unique complex structure on D such that the projection $p_D \colon D \to X$ is a locally biholomorphic map.

A domain (D, p_D) is said to be contained in another domain (G, p_G) if there is a map $j \colon D \to G$ such that $p_G \circ j = p_D$. We note that the "inclusion" j is a priori only locally biholomorphic and need not be globally injective.

For instance, every ordinary domain (that is, connected open subset) $D \subset X$ can be regarded as a domain over X by setting $p_D = \mathrm{id}$. In this case, the map $j \colon D \to G$ is injective because $p_G \circ j = \mathrm{id}$.

A domain (D, p_D) is said to be *locally Stein* if every point $x \in X$ has a neighbourhood $V \ni x$ whose pre-image $p_D^{-1}(V) \subset D$ is a Stein manifold.

2.2. Envelopes of holomorphy. The *envelope of holomorphy* of a domain (D, p_D) over X is the maximal domain $(H(D), p_{H(D)})$ over the same manifold X such that every holomorphic function on D extends to a holomorphic function on $H(D)$.

More precisely, this means that we are given a locally biholomorphic map $\alpha = \alpha_D \colon D \to H(D)$ such that the following conditions hold:

1) inclusion: $p_{H(D)} \circ \alpha = p_D$;

2) extension: for every holomorphic function $f \in \mathcal{O}(D)$ there is a holomorphic function $F \in \mathcal{O}(H(D))$ such that $F \circ \alpha = f$;

3) maximality: if a domain $p_G \colon G \to X$ and a locally biholomorphic map $\beta \colon D \to G$ satisfy 1) and 2) with $H(D)$ replaced by G, then there is a locally biholomorphic map $\gamma \colon G \to H(D)$ such that $\gamma \circ \beta = \alpha$ and $p_{H(D)} \circ \gamma = p_G$.

The envelope of holomorphy exists and is unique (up to a natural isomorphism) by Thullen's theorem. We note once again that the map $\alpha \colon D \to H(D)$ may be non-injective. However, it is injective if $D \subset X$ is an ordinary domain in X.

Cartan–Thullen and Oka showed that $(H(D), p_{H(D)})$ is a *locally* Stein domain over X. Oka and Docquier–Grauert proved that every locally Stein domain over a Stein manifold is Stein. It follows that *the envelope of holomorphy of any domain over a Stein manifold is a Stein manifold*.

Another useful observation is that holomorphic maps to Stein manifolds extend in the same way as holomorphic functions.

Lemma 2.1. *Any holomorphic map $f\colon D \to Y$ to a Stein manifold extends to a holomorphic map $F\colon H(D) \to Y$ (in the sense that $F \circ \alpha_D = f$, where $\alpha_D\colon D \to H(D)$ is the natural map to the envelope of holomorphy).*

Proof. This is almost obvious. Let $i\colon Y \to \mathbb{C}^N$ be a proper holomorphic embedding of Y into complex Euclidean space of sufficiently high dimension. The components of the map $i \circ f\colon D \to \mathbb{C}^N$ are holomorphic functions on D and therefore extend to $H(D)$. The image of the extended map is contained in $Y = i(Y)$ by the uniqueness theorem.

2.3. Coverings and envelopes over Stein manifolds. Let (D, p_D) be a domain over a Stein manifold X. Let $\pi\colon \widehat{D} \to D$ be the universal covering of D. Then the pair $(\widehat{D}, p_{\widehat{D}})$ with $p_{\widehat{D}} \stackrel{\text{def}}{=} p_D \circ \pi$ is a domain over X. Hence we can consider its envelope of holomorphy $(H(\widehat{D}), p_{H(\widehat{D})})$.

Theorem 2.2 [16]. *The envelope of holomorphy of the universal covering of D coincides with the universal covering of the envelope of holomorphy of D. More precisely, there is a commutative diagram of locally biholomorphic maps*

$$\begin{array}{ccc} \widehat{D} & \xrightarrow{\alpha_{\widehat{D}}} & H(\widehat{D}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle H(\pi)} \\ D & \xrightarrow{\alpha_D} & H(D) \end{array}$$

where the horizontal arrows are the natural maps into the envelopes of holomorphy and the vertical arrows are the universal coverings.

It may be helpful to bear in mind the following interpretation of this theorem in terms of the Weierstrass theory of analytic continuation. A holomorphic function on a covering of D corresponds to a germ of a holomorphic function that can be extended analytically along every path in D. (From this point of view, a covering is a domain over D without boundary points. See [11] for the definition of boundary points of Riemann domains.) Theorem 2.2 may be restated as follows. *If a germ of a holomorphic function can be extended along every path in D, then it can also be extended along every path in the envelope of holomorphy $H(D)$.*

Example 2.3 (enhanced monodromy theorem). The classical monodromy theorem states that if a germ of a holomorphic function can be extended unboundedly in a simply connected domain, then this extension is single-valued (and hence defines a holomorphic function). It follows from Theorem 2.2 that the same conclusion holds if we assume only that the envelope of holomorphy of the domain is simply connected.

Example 2.4 (pseudoconvex domains). Let $D \subset X$ be a (weakly) pseudoconvex domain with smooth boundary in a Stein manifold X of complex dimension $n \geqslant 2$. Then D is Stein by the Oka–Docquier–Grauert theorem. Let $V \subset D$ be the intersection of a tubular neighbourhood of the boundary of D in X with the domain D. It is an open subset of D and is homotopy equivalent to the boundary ∂D. Every holomorphic function on V extends to the whole of D by Hartogs' theorem applied to D. It follows that V is connected, and the envelope of holomorphy of V is precisely D.

Theorem 2.2 shows that the envelope of holomorphy of the universal covering $\pi\colon \widehat{V} \to V$ is the universal covering of D. In other words, if a germ of a holomorphic function can be extended along every path in V, then it extends along every path in D. In this case, the enhanced monodromy theorem asserts that if D is simply connected, then every holomorphic function $f \in \mathcal{O}(\widehat{V})$ is the pullback $F \circ \pi$ of some holomorphic function $F \in \mathcal{O}(D)$.

In view of this observation, it is worthwhile to compare the fundamental groups of D and V or, equivalently, of D and ∂D. A standard application of Morse theory to Stein manifolds (as in [2] or [3]) shows that the homomorphism $\pi_1(V) \to \pi_1(D)$ is an isomorphism if the complex dimension $n \geqslant 3$. Hence we get no improvement on the ordinary monodromy theorem in this case.

If $n = 2$, then the homomorphism $\pi_1(V) \to \pi_1(D)$ is only surjective, and the fundamental group of the boundary can be much larger than that of the domain. For instance, tom Dieck and Petrie [6] gave explicit examples of affine algebraic surfaces $\Sigma \subset \mathbb{C}^3$ such that the intersection of Σ with a sufficiently large ball is a *contractible* strictly pseudoconvex domain whose boundary has an *infinite* fundamental group.

2.4. Envelopes over complex projective space. Analytic continuation over $\mathbb{CP}^n$ can be understood quite well with the help of the following theorem of Fujita [9] and Takeuchi [23]. *A locally Stein domain over* $\mathbb{CP}^n$ *either is Stein or coincides with* $\mathbb{CP}^n$. An elegant and illuminating proof of this result was given by Ueda [24].

In particular, for any domain (D, p_D) over $\mathbb{CP}^n$, its envelope of holomorphy $H(D)$ is either a Stein domain over $\mathbb{CP}^n$ or the tautological domain $(\mathbb{CP}^n, \mathrm{id})$. Clearly, the latter option can be characterized by the property that every holomorphic function on D is constant.

Theorem 2.5. *Theorem* 2.2 *holds for every domain* (D, p_D) *over the complex projective space* $\mathbb{CP}^n$.

Proof. If the envelope $H(D)$ is Stein, then we can equip D with the locally biholomorphic map $\alpha_D\colon D \to H(D)$ and regard it as a domain over $H(D)$. Consequently, the result follows directly from Kerner's theorem in this case.

If $H(D) = \mathbb{CP}^n$, then the analogue of Theorem 2.2 is given by the following proposition.

Proposition 2.6. *Let* (D, p_D) *be a domain over* $\mathbb{CP}^n$ *such that every holomorphic function on* D *is constant. Then the same holds for every covering* $\pi\colon \widehat{D} \to D$.

Proof. It suffices to consider the case of the universal covering $\pi\colon \widehat{D} \to D$ because every holomorphic function on any covering of D can be pulled back to it.

We assume that $H(\widehat{D})$ is Stein and seek a contradiction by imitating Kerner's argument in [16], pp. 127–129. Let Δ be the group of deck transformations of $\widehat{D}$. The action of this group on $\widehat{D}$ extends to a properly discontinuous free action of a quotient group $\widetilde{\Delta}$ on $H(\widehat{D})$. The quotient $H(\widehat{D})/\widetilde{\Delta}$ is a domain over $\mathbb{CP}^n$. It contains $D = \widehat{D}/\Delta$ and is covered by $H(\widehat{D})$. Lemma 1 (Hilfssatz 1) of [16] shows that this domain is *locally* Stein. However, it cannot be Stein because it would then follow that there are non-constant holomorphic functions on D. Hence $H(\widehat{D})/\widetilde{\Delta} = \mathbb{CP}^n$ by Fujita's theorem. Since $\mathbb{CP}^n$ is simply connected, it follows that $H(\widehat{D}) = \mathbb{CP}^n$, a contradiction.

To consider extensions of meromorphic functions, we introduce the *envelope of meromorphy* $(M(D), p_{M(D)})$ of a domain (D, p_D) over a complex manifold X. The definition is entirely analogous to the holomorphic case. Namely, $(M(D), p_{M(D)})$ is the maximal domain over X containing (D, p_D) and such that every meromorphic function on D extends meromorphically to $M(D)$. The existence and uniqueness of the envelope of meromorphy follow from a Thullen-type theorem. Levi's theorem on the extension of meromorphic functions and the ubiquitous theorem of Oka imply that the envelope of meromorphy is a locally Stein domain over X.

One can show that the envelope of meromorphy of any domain over complex projective space coincides with its envelope of holomorphy, but we shall only need the following partial result.

Proposition 2.7. *Let (D, p_D) be a domain over $\mathbb{CP}^n$ such that every holomorphic function on D is constant. Then every meromorphic function on D has the form $f \circ p_D$ for some rational function f on $\mathbb{CP}^n$.*

Proof. We must show that the envelope of meromorphy of (D, p_D) coincides with $\mathbb{CP}^n$. Then every meromorphic function on D can be obtained as the pullback of a meromorphic function on $\mathbb{CP}^n$, which must be rational by Serre's GAGA principle.

Suppose that the envelope of meromorphy is not $\mathbb{CP}^n$. Since it is locally Stein, it must be a Stein domain by Fujita's theorem. But this implies that (D, p_D) is contained in a Stein domain and hence admits non-constant holomorphic functions, a contradiction.

Example 2.8 (hypersurfaces with indefinite Levi form in $\mathbb{CP}^n$). Let $M \subset \mathbb{CP}^n$ be a smooth compact real hypersurface whose Levi form is non-degenerate and indefinite (that is, has positive and negative eigenvalues at each point). We consider a connected neighbourhood $U \supset M$ and denote its envelopes of holomorphy and meromorphy by $H(U)$ and $M(U)$ respectively. Note that neither envelope is Stein because the hypersurface M cannot lie in a Stein manifold. Hence $H(U) = M(U) = \mathbb{CP}^n$ by Fujita's theorem. In other words, every holomorphic function on U is constant and every meromorphic function on U is rational.

Furthermore, let $\pi\colon \widehat{U} \to U$ be a covering of U. Then every holomorphic function on $\widehat{U}$ is constant by Proposition 2.6 and therefore every meromorphic function on $\widehat{U}$ is the pullback of a rational function by Proposition 2.7. Thus $H(\widehat{U}) = M(\widehat{U}) = \mathbb{CP}^n$.

2.5. Extension of locally biholomorphic maps. The theory of envelopes of holomorphy and meromorphy (as recalled in this section) enables us to give a unified treatment (and a slight generalization) of the result on the holomorphic extension of locally biholomorphic maps established by Kerner [15] and Ivashkovich [13].

Theorem 2.9. *Suppose that X and Y are complex manifolds and each of them is either Stein or biholomorphic to the complex projective space $\mathbb{CP}^n$. Let (D, p_D) be a domain over X and let $(H(D), p_{H(D)})$ be its envelope of holomorphy.*

Then every locally biholomorphic map $f\colon D \to Y$ extends to a locally biholomorphic map $F\colon H(D) \to Y$.

Proof. We begin with a simple general observation. Suppose that $A \subset H(D)$ is a non-empty complex hypersurface (a complex analytic subset of pure codimension one). Then A must intersect the image $\alpha_D(D)$ of the domain D in its envelope of holomorphy. Indeed, the complement $H(D)\setminus A$ is a proper locally Stein open subset of $H(D)$ and hence must be Stein by the theorems of Oka–Docquier–Grauert (if $H(D)$ is Stein) and Fujita (if $H(D) = \mathbb{CP}^n$). It follows that there is a holomorphic function (say, $g \in \mathcal{O}(H(D) \setminus A)$) which cannot be extended to $H(D)$. If $\alpha_D(D)\cap A$ were empty, then the function $g\circ\alpha_D \in \mathcal{O}(D)$ would not extend to $H(D)$, contradicting the definition of the envelope of holomorphy.

Now assume that the target manifold Y is Stein. Then the map $f\colon D \to Y$ extends to a holomorphic map $F\colon H(D) \to Y$ by Lemma 2.1. Consider the ramification locus of F, that is, the complex hypersurface in $H(D)$ consisting of the points at which $\operatorname{rank}_{\mathbb{C}}(F) < n = \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$. This hypersurface cannot intersect $\alpha_D(D)$, and so it is empty by the observation above. Thus F is locally biholomorphic.

The case $Y = \mathbb{CP}^n$ splits into two subcases. Suppose first that D admits a non-constant holomorphic function. Since the map $f\colon D \to \mathbb{CP}^n$ is locally biholomorphic, we can regard the pair (D, f) as a domain over Y. Let $(\widetilde{D}, \tilde{f}\,)$ be the envelope of holomorphy of this domain over Y. (We use a different notation for envelopes over Y to avoid confusion.) Let $\beta\colon D \to \widetilde{D}$ be the natural map into the envelope. We recall that $\tilde{f} \circ \beta = f$. Since $\mathcal{O}(D) \neq \mathbb{C}$, the envelope $\widetilde{D}$ is a Stein manifold by Fujita's theorem. As we have already seen, the map $\beta\colon D \to \widetilde{D}$ extends to a locally biholomorphic map $\mathrm{B}\colon H(D) \to \widetilde{D}$ such that $\mathrm{B} \circ \alpha_D = \beta$. The composite $F \overset{\text{def}}{=} \tilde{f} \circ \mathrm{B}$ is the desired extension of f to $H(D)$. Indeed,

$$F \circ \alpha_D = \tilde{f} \circ \mathrm{B} \circ \alpha_D = \tilde{f} \circ \beta = f.$$

Finally, consider the case when $\mathcal{O}(D) = \mathbb{C}$. This is possible only if $X = \mathbb{CP}^n$. Thus we can apply Proposition 2.7 to the components of the map f with respect to an affine coordinate system on $Y = \mathbb{CP}^n$. It follows that there is a rational map $F\colon \mathbb{CP}^n \to \mathbb{CP}^n$ such that $F \circ \alpha_D = f$. Arguing in the same way as before, we see that the ramification locus of F (defined to be the Zariski closure of the set of points at which F is holomorphic and not of maximal rank) must be empty. Therefore the map F is locally biholomorphic on the complement of its indeterminacy locus I. However, I has complex codimension at least 2, so we can apply the preceding case of the theorem locally, in a Stein neighbourhood of each point of I, and conclude that F is in fact locally biholomorphic everywhere.

Notice that a locally biholomorphic map from $\mathbb{CP}^n$ to itself is just a linear automorphism. Hence we get the following corollary (compare [13] and [12]).

Corollary 2.10. *Let (D, p_D) be a domain over $\mathbb{CP}^n$ such that every holomorphic function on D is constant. Then every locally biholomorphic map $f\colon D \to \mathbb{CP}^n$ has the form $L \circ p_D$ for a linear automorphism $L \in \operatorname{Aut}(\mathbb{CP}^n)$.*

The proof of Theorem 2.9 rightly suggests that a holomorphic but not locally biholomorphic map $f\colon D \to \mathbb{CP}^n$ may admit no holomorphic extension to $H(D)$.

For instance, the quadratic transformation $Q\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ is a birational map with three indeterminacy points. Clearly, if D is a punctured neighbourhood of one of these points, then Q is holomorphic on D but cannot be holomorphically extended to $H(D)$. This phenomenon was discovered by Ivashkovich [14] and became the starting point for his deep results on meromorphic continuation (see, for example, [14]).

Remark 2.11. Ueda [24] generalized Fujita's theorem to Grassmann manifolds $\mathrm{Gr}_{\mathbb{C}}(m,n)$. Hence the same proof shows that Theorem 2.9 and Corollary 2.10 remain valid if we replace $\mathbb{CP}^n$ by any complex Grassmannian.

§ 3. Analytic continuation along real hypersurfaces

3.1. Extension of local equivalences between real hypersurfaces. Two real hypersurfaces M and M' in $\mathbb{C}^n$ are said to be *locally biholomorphically equivalent* (or simply *equivalent*) at points $p \in M$ and $p' \in M'$ if there are connected open neighbourhoods $U \ni p$ and $U' \ni p'$ in $\mathbb{C}^n$ and a biholomorphic map $f\colon U \to U'$ such that $f(M \cap U) = M' \cap U'$ and $f(p) = p'$. An important feature of local equivalences between real-analytic hypersurfaces in several complex variables is the property of propagation along paths. The first result of this type was obtained by Pinchuk.

Theorem 3.1 ([18]). *Let $M \subset \mathbb{C}^n$ be a connected strictly pseudoconvex real-analytic hypersurface which is equivalent to the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ at some point $p \in M$. Then the germ of the equivalence map ${}_p\mathbf{f}$ extends along any path on M as a locally biholomorphic map sending M to S^{2n-1}.*

This result enables us to define *spherical* hypersurfaces as being connected, strictly pseudoconvex, real-analytic and equivalent to S^{2n-1} at some (and therefore every) point. It is an immediate corollary of Pinchuk's result that a compact simply connected spherical hypersurface M in an arbitrary complex manifold X is biholomorphic to the standard sphere. Hence Theorem 3.1 can be formulated with the sphere replaced by any compact simply connected spherical hypersurface M'. The result will be false, however, if M' is only assumed to be spherical and compact (see [4]).

Pinchuk proved in [19] that the compactness assumption suffices in the *non-spherical* case, that is, when S^{2n-1} is replaced by any compact strictly pseudoconvex real-analytic hypersurface $M' \subset \mathbb{C}^n$ which is not equivalent to the sphere at any point. This was generalized to hypersurfaces in arbitrary complex manifolds in [25].

Theorem 3.2 ([20], [25], [26]). *Let X and X' be complex manifolds with* $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} X' = n$. *Suppose that $M \subset X$ and $M' \subset X'$ are real-analytic strictly pseudoconvex non-spherical hypersurfaces with M connected and M' compact. If M and M' are locally equivalent at points $p \in M$ and $p' \in M'$, then the germ ${}_p\mathbf{f}$ of the equivalence map extends along any path on M as a locally biholomorphic map sending M to M'.*

Much less is known about maps between hypersurfaces which are not strictly pseudoconvex. Beloshapka and Ezhov have given examples suggesting that an extension of Theorem 3.2 to this situation may be problematic. On the other hand, the reflection principle underlying the proof of Theorem 3.1 can still be used in its geometric form provided that the target hypersurface is real-algebraic (see [21], [22]). This approach yields the following result for hypersurfaces with non-degenerate indefinite Levi form.

Theorem 3.3 ([12]). *Let M be a connected real-analytic Levi non-degenerate hypersurface in a complex manifold X and let M' be a compact real-algebraic Levi non-degenerate hypersurface in $\mathbb{CP}^n$. Suppose that M and M' are locally equivalent at points $p \in M$ and $p' \in M'$. Then the germ ${}_p\mathbf{f}$ of the equivalence map extends along any path on M as a locally biholomorphic map sending M to M'.*

This theorem was applied in [12] to prove Theorem B for a simply connected hypersurface $M \subset \mathbb{CP}^n$. We also note that if M and M' are real-algebraic, then Theorem 3.3 essentially follows from a well-known theorem of Webster [27], which states that if M and M' are real-algebraic hypersurfaces in $\mathbb{C}^n$ and are locally equivalent at Levi non-degenerate points $p \in M$ and $p' \in M'$, then the equivalence map is algebraic, that is, its graph is contained in an algebraic subvariety of $\mathbb{C}^n \times \mathbb{C}^n$.

3.2. Analytic continuation to open neighbourhoods. If M is not simply connected, then the analytic continuation of a germ ${}_p\mathbf{f}\colon M \to M'$ along non-homotopic paths starting at $p \in M$ with the same endpoint $q \in M$ may produce different extensions, which a priori may have different radii of convergence at q. This cannot happen for maps between non-spherical strictly pseudoconvex hypersurfaces by a result of Vitushkin (see [26], § 8.1). In general, an extension to a certain fixed open set can be obtained using the theory presented in § 2.

Lemma 3.4. *Let $M \subset X$ be a compact strictly pseudoconvex hypersurface and let D' be a strictly pseudoconvex Stein domain. Suppose that a germ ${}_p\mathbf{f}\colon M \to \partial D'$ of a locally biholomorphic map extends along any path in M as a locally biholomorphic map sending M to $\partial D'$. Then there is a neighbourhood $U \subset X$ of M such that ${}_p\mathbf{f}$ extends as a locally biholomorphic map with values in D' along any path in U^-, the strictly pseudoconvex one-sided neighbourhood of M.*

Proof. It suffices to prove that every point $q \in M$ has a neighbourhood $U_q \subset X$ such that any holomorphic map g obtained by analytic continuation of ${}_p\mathbf{f}$ extends to the strictly pseudoconvex side of U_q. Let V be a neighbourhood of q such that $M \cap V$ is simply connected. Then g extends to a locally biholomorphic map from some neighbourhood of $M \cap V$. This extension takes values in D' on the pseudoconvex side of M. Since M is strictly pseudoconvex, there is a neighbourhood $U_q \subset X$ of q such that every function holomorphic in a one-sided neighbourhood of $M \cap V$ extends holomorphically to U_q^-, the pseudoconvex side of U_q. The same holds for locally biholomorphic maps to the Stein manifold D' by Theorem 2.9. By construction, U_q is independent of g.

Lemma 3.5 ([13]). *Let $M \subset X$ be a compact real hypersurface with non-degenerate indefinite Levi form. Suppose that a germ ${}_p\mathbf{f}\colon M \to \mathbb{CP}^n$ of a locally biholomorphic map to complex projective space extends as a locally biholomorphic map along any path on M. Then there is a neighbourhood $U \subset X$ of M such that ${}_p\mathbf{f}$ extends as a locally biholomorphic map along any path in U.*

Proof. The argument is very similar to the previous one. For a point $q \in M$, let $V \ni q$ be a coordinate neighbourhood such that $M \cap V$ is simply connected. By a theorem of Hans Lewy, there is a neighbourhood $U_q \subset V$ of q such that every function holomorphic in a neighbourhood of $M \cap V$ extends holomorphically to U_q. By Theorem 2.9, the same extension property holds for locally biholomorphic maps to $\mathbb{CP}^n$. Hence any holomorphic map obtained by analytic continuation of ${}_p\mathbf{f}$ extends to a locally biholomorphic map $U_q \to \mathbb{CP}^n$.

§ 4. Global extension of local maps

4.1. The general set-up and proof of Theorem A in the non-spherical case. Let D and D' be strictly pseudoconvex Stein domains with real-analytic boundaries. Suppose that ∂D and $\partial D'$ are locally equivalent at points $p \in \partial D$ and $p' \in \partial D'$. Let ${}_p\mathbf{f}\colon \partial D \to \partial D'$ be the germ of a locally biholomorphic map realizing this equivalence.

Proposition 4.1. *If the germ ${}_p\mathbf{f}$ can be extended as a locally biholomorphic map sending ∂D to $\partial D'$ along any path in ∂D, then it can be extended as a locally biholomorphic map with values in $\overline{D'}$ along any path in $\overline{D}$.*

Proof. By Lemma 3.4, there is a neighbourhood $U \supset \partial D$ such that the germ ${}_p\mathbf{f}$ can be extended along any path in $V = U \cap D$ as a locally biholomorphic map with values in D'. This extension defines a locally biholomorphic map $f\colon \widehat{V} \to D'$ from the universal covering $\widehat{V} \to V$.

The envelope of holomorphy of V is precisely D by Hartogs' theorem (see Example 2.4). Hence the envelope of holomorphy of the universal covering $\widehat{V} \to V$ is the universal covering $\pi\colon \widehat{D} \to D$ by Kerner's theorem. Then Theorem 2.9 shows that the map $f\colon \widehat{V} \to D'$ extends to a locally biholomorphic map $F\colon \widehat{D} \to D'$.

Let $\overline{\pi}\colon \overline{Y} \to \overline{D}$ be the universal covering of the closure of D. Then $\overline{Y}$ is a complex manifold with (not necessary compact) boundary $\partial Y = \overline{\pi}^{-1}(\partial D)$ and interior $Y = \overline{Y} - \partial Y = \widehat{D}$. By construction, the map $F\colon Y \to D'$ coincides with a lift of an extension of ${}_p\mathbf{f}$ near every boundary point $q \in \partial Y$. Hence it extends to a locally biholomorphic map $\overline{F}\colon \overline{Y} \to \overline{D'}$ of complex manifolds with boundary. In other words, the germ ${}_p\mathbf{f}$ extends along any path in $\overline{D}$ as a locally biholomorphic map with values in $\overline{D'}$.

We are now in a position to prove the stronger form of Theorem A for non-spherical domains (Theorem A.1). Indeed, let ${}_p\mathbf{f}\colon \partial D \to \partial D'$ be a local equivalence germ. By Theorem 3.2, this germ extends along any path in ∂D. Therefore it extends as a locally biholomorphic map along any path in $\overline{D}$ by Proposition 4.1. The same conclusion holds for the inverse map ${}_{p'}\mathbf{f}^{-1}\colon \partial D' \to \partial D$. Hence it follows from the monodromy theorem that the extension of ${}_p\mathbf{f}$ determines a biholomorphism of the universal covering of the closure of D onto the universal covering of the closure of D'.

Corollary 4.2. *Suppose that D and D' are strictly pseudoconvex Stein domains with locally equivalent non-spherical boundaries. If D has finite fundamental group, then so does D'.*

Proof. The fundamental group of a compact manifold with boundary (for example, $\overline{D}$) is finite if and only if the universal covering of this manifold is compact.

4.2. Uniformization of Stein domains with spherical boundary. Any local equivalence between spherical hypersurfaces factors through the sphere. Therefore it suffices to prove Theorem A in the case when D' is the unit ball $B \subset \mathbb{C}^n$. We denote the unit sphere by $S = \partial B$. By Theorem 3.1 and Proposition 4.1, any local equivalence germ ${}_p\mathbf{f}\colon \partial D \to S$ extends to a locally biholomorphic map $\overline{F}\colon \overline{Y} \to \overline{B}$ (of complex manifolds with boundary) from the universal covering of $\overline{D}$ to the closed unit ball.

The map $\overline{F}$ may be viewed as the extension of the "developing map" of the boundary (introduced by Burns and Shnider [4]) to the universal covering of the domain. In particular, it inherits the following important equivariance property.

Lemma 4.3. *Let $\Gamma = \pi_1(D) = \pi_1(\overline{D}\,)$ be the group of deck transformations of the universal covering $\overline{\pi}\colon \overline{Y} \to \overline{D}$. There is a representation $\rho\colon \Gamma \to \operatorname{Aut}(B)$ such that $\rho(\gamma) \circ \overline{F}(x) = \overline{F} \circ \gamma(x)$ for all $x \in \overline{Y}$ and $\gamma \in \Gamma$.*

Proof. The existence of a representation ρ such that the required relation holds for all $x \in \partial Y$ is an immediate corollary of the Poincaré–Alexander theorem [1] and was observed by Burns and Shnider ([4], § 1). The extension to the whole of $\overline{Y}$ follows by the uniqueness theorem.

Examples in [4] show that the inverse germ ${}_{p'}\mathbf{f}^{-1}\colon S \to \partial D$ may admit no extension along any path in S. On the other hand, the inverse map extends along every path in the *open* ball B. To see this, it is enough to prove the following assertion.

Lemma 4.4. *There is $\varepsilon > 0$ such that every point $x \in Y$ has an open neighbourhood $V \subset Y$ with the following properties*:

1) *the restriction $F|_V\colon V \to F(V)$ is biholomorphic*,

2) *$F(V)$ contains the ball of radius ε centred at $F(x)$ with respect to the Poincaré metric on B.*

Proof. Let h be the Euclidean metric on $\mathbb{C}^n$. The Poincaré metric dominates the Euclidean metric in the ball B. In particular, the Euclidean ball of radius $R > 0$ centred at a point $b \in B$ contains the Poincaré ball of the same radius centred at b.

We denote by $\overline{F}^*h$ the pullback of the Euclidean metric to the manifold $\overline{Y}$. Let $\Phi \subset Y$ be a fundamental domain for the action of Γ on Y. We note that Φ is a relatively compact subset of $\overline{Y}$. It follows that there is $\varepsilon > 0$ such that, for every point $x \in \Phi$, the map $\overline{F}$ is a biholomorphism of the ball of radius ε centred at x with respect to the metric $\overline{F}^*h$ onto the intersection of the Euclidean ball of the same radius centred at $\overline{F}(x)$ with the closed unit ball $\overline{B}$. Since the Euclidean ball about an interior point of B contains the Poincaré ball with the same centre and radius, we see that every point of Φ has a neighbourhood with properties 1) and 2).

Let $x \in Y$ be an arbitrary point. By the definition of fundamental domain, there is a deck transformation $\gamma \in \Gamma$ such that $\gamma(x) \in \Phi$. Let W be the neighbourhood of $\gamma(x)$ constructed above. We put $V = \gamma^{-1}(W)$. By Lemma 4.3, we have

$$F = \rho(\gamma)^{-1} \circ F \circ \gamma.$$

It follows that F is biholomorphic in V if and only if it is biholomorphic in W. Furthermore, the image $F(V) = \rho(\gamma)^{-1}(F(W))$ contains the Poincaré ball of radius ε about $F(x)$ because $F(W)$ contains the ball of this radius about $F(\gamma(x))$ while the automorphism $\rho(\gamma)^{-1}$ is an isometry of the Poincaré metric.

Since B is simply connected, it follows that the map $F\colon Y \to B$ of open manifolds is biholomorphic. This completes the proofs of Theorems A and A.2.

Remark 4.5 (boundary behaviour: I). Once F is known to be biholomorphic, it is easy to see that $\overline{F}$ is injective on ∂Y. Hence $\overline{F}$ is a biholomorphic map of $\overline{Y}$ onto $\overline{B} \setminus A$, where $A = S \setminus \overline{F}(\partial Y)$ is a closed subset of the unit sphere. In other words, the closure of the domain D is uniformized by $\overline{B} \setminus A$. However, the subset A depends on D, and its structure remains a mystery.

The uniformization theorem imposes strong restrictions on the topology of spherical domains.

Corollary 4.6. *Let D be a strictly pseudoconvex Stein domain with spherical boundary. Then the higher homotopy groups $\pi_k(D)$ are trivial for all $k \geqslant 2$. If D is not biholomorphic to the ball, then its fundamental group is infinite and contains no non-trivial finite subgroups.*

Proof. Let us give a purely topological proof of this fact. The higher homotopy groups vanish because the universal covering of D is contractible. Suppose that $\pi_1(D)$ contains a non-trivial element of finite prime order $p > 0$. Let $\widetilde{D}$ be the covering of D corresponding to the subgroup generated by this element. Then $\pi_1(\widetilde{D}) = \mathbb{Z}/p\mathbb{Z}$ and $\pi_k(\widetilde{D}) = 0$ for all $k \geqslant 2$. By the Hurewicz–Eilenberg–McLane theorem, the cohomology groups of $\widetilde{D}$ with any coefficients are isomorphic to the cohomology groups of its fundamental group. However, $H^k(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) \neq 0$ for all $k \geqslant 0$ (see, for example, [7], p. 28) while the space $\widetilde{D}$, which is a finite-dimensional manifold, cannot have non-trivial cohomology in all positive dimensions. This contradiction shows that every finite subgroup of $\pi_1(D)$ is trivial.

Remark 4.7 (boundary behaviour: II). The Lefschetz theorem for Stein manifolds [2], [3] tells us that the homomorphism $\pi_k(\partial D) \to \pi_k(D)$ is an isomorphism for all $k \leqslant n-2$ and is surjective for $k = n-1$, where $n = \dim_{\mathbb{C}} D$. In particular, $\pi_1(\partial D)$ surjects onto $\pi_1(D)$ for all $n \geqslant 2$, and we retrieve the result of Burns and Shnider [4] that a compact spherical hypersurface bounding a Stein domain other than the ball must have an infinite fundamental group.

If the complex dimension $n \geqslant 3$, then $\pi_1(\partial D) = \pi_1(D)$. It follows that, firstly, $\pi_1(\partial D)$ does not possess non-trivial finite subgroups and, secondly, the covering $\partial Y \to \partial D$ is the universal covering of the boundary. On the other hand, the paper [10] provides examples of strictly pseudoconvex Stein quotients of the unit ball in $\mathbb{C}^2$ having torsion elements in $\pi_1(\partial D)$. The same examples show that the covering $\partial Y \to \partial D$ is generally not the universal covering of the boundary in the two-dimensional case.

4.3. Proof of Theorem B. The argument follows the familiar pattern. Let ${}_p\mathbf{f}\colon M \to M'$ be the germ of a local equivalence between real-analytic compact hypersurfaces with non-degenerate indefinite Levi form in $\mathbb{CP}^n$. If M' is real-algebraic, it follows from Theorem 3.3 that this germ extends as a locally biholomorphic map along any path in M. So it remains to prove the following generalization of Corollary 2 in [13] to the case of multi-valued maps.

Proposition 4.8. *Let $M \subset \mathbb{CP}^n$ be a compact real hypersurface with non-degenerate indefinite Levi form. If the germ of a locally biholomorphic map ${}_p\mathbf{f}\colon M \to \mathbb{CP}^n$ extends as a locally biholomorphic map along every path in M, then ${}_p\mathbf{f} = {}_p\mathbf{L}$ for some automorphism $L \in \mathrm{Aut}(\mathbb{CP}^n)$.*

Proof. The germ ${}_p\mathbf{f}$ extends as a locally biholomorphic map along any path in a neighbourhood $U \supset M$ provided by Lemma 3.5. This extension determines a locally biholomorphic map $F\colon \widehat{U} \to \mathbb{CP}^n$ from the universal covering of the neighbourhood U.

We are now in the situation discussed in Example 2.8. In particular, we know that every holomorphic function on $\widehat{U}$ is constant. Hence Corollary 2.10 shows that the map $F\colon \widehat{U} \to \mathbb{CP}^n$ is the pullback of an automorphism $L \in \mathrm{Aut}(\mathbb{CP}^n)$. This is equivalent to the assertion about the germs.

Example 4.9. Let us briefly outline a construction showing that there are many topologically different examples of real-algebraic hypersurfaces with non-degenerate indefinite Levi form in $\mathbb{CP}^n$. It suffices to exhibit smooth Levi non-degenerate hypersurfaces because they can be approximated by real-algebraic hypersurfaces which thus have the same signature of the Levi form and the same topology.

Let $Z \subset \mathbb{CP}^n$ be a complex submanifold of dimension $k \geqslant 0$. A well-known result (going back to Grauert) states that, since the normal bundle of Z is positive, the Levi form of the boundary $M = \partial U$ of an appropriate tubular neighbourhood $U \supset Z$ has signature $(n-k-1, k)$. Here the $n-1-k$ positive directions are "perpendicular" to Z, and the k negative directions are "parallel" to Z. For instance, a point ($k=0$) has a strictly pseudoconvex neighbourhood, and a complex hypersurface ($k = n-1$) has a strictly pseudoconcave one. The standard real hyperquadrics can be obtained using this construction from linear subspaces $Z \subset \mathbb{CP}^n$ of appropriate (co)dimension.

Topologically, M is an $S^{2(n-k)-1}$-bundle over Z. Thus the fundamental group of M is isomorphic to that of Z if the complex codimension of Z is at least 2. For instance, if Z is a complex curve of genus $g > 0$ in $\mathbb{CP}^3$, then M is a real hypersurface of Levi signature $(1,1)$ with infinite fundamental group.

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Received 10/MAR/05
Translated by S. YU. NEMIROVSKII

Typeset by $\mathcal{AMS}$-TEX