EXTENSION OF HOLOMORPHIC MAPS BETWEEN REAL HYPERSURFACES OF DIFFERENT DIMENSION

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Abstract. It is proved that a germ of a holomorphic map from a real analytic hypersurface M in $\mathbb{C}^n$ into a strictly pseudoconvex compact real algebraic hypersurface M' in $\mathbb{C}^N$, $1 < n \le N$ extends holomorphically along any path on M.

1. Introduction

In this paper we consider the problem of analytic continuation of a germ of a holomorphic map sending a real analytic hypersurface into another such hypersurface in the special case when the target hypersurface is real algebraic but of higher dimension. Our principal result is the following.

Theorem 1.1. *Let M be a connected smooth real-analytic minimal hypersurface in $\mathbb{C}^n$, M' be a compact strictly pseudoconvex real algebraic hypersurface in $\mathbb{C}^N$, $1 < n \le N$. Suppose that f is a germ of a holomorphic map at a point $p \in M$ and $f(M) \subset M'$. Then f extends as a holomorphic map along any smooth CR-curve on M with the extension sending M to M'.*

We note that when M is a minimal hypersurface, then the CR-orbit of p is all of M (see Section 2 for details), and therefore, the theorem above gives analytic continuation of f to every point of M.

In the equidimensional case the problem of analytic continuation of a germ of a map between real analytic hypersurfaces has attracted a lot of attention (see, for example, [24], [35], [34], [26], [30] and [32]). This problem, which originated in the work of Poincaré [28] (generalized later in [33] and [1]), is related to other fundamental questions in several complex variables, such as boundary regularity of proper holomorphic mappings, the theory of CR maps, and classification of domains in complex spaces (for the latter connection see [34], [26], [32], [23]).

The situation seems to be more delicate in the case of different dimensions. The first result of this type is probably due to Pinchuk [25] who proved that a germ of a holomorphic map from a strictly pseudoconvex real analytic hypersurface $M \subset \mathbb{C}^n$ into a sphere S^{2N-1}, $1 < n \le N$, extends holomorphically along any path on M. Just recently Diederich and Sukhov [9] proved that the same extension holds if M is weakly pseudoconvex. Theorem 1.1 is a direct generalization of these results (although our methods are quite different). Further, in the case when $\dim M = \dim M'$, Theorem 1.1 generalizes the result in [30], where the hypersurface M was assumed to be essentially finite, a stronger condition than minimality. Other related results also include various extensions obtained when both M and M' are algebraic (see e.g. [16], [29], [2], [6], [36], [19] and references therein), which state that under certain conditions a map between two real algebraic submanifolds (or even sets) is algebraic, and therefore extends to a dense open subset of $\mathbb{C}^n$.

Much like in the equidimensional case, analytic continuation can be used to prove boundary regularity of holomorphic maps.

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Theorem 1.2. *Let D and D' be smoothly bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively, $1 < n \le N$, ∂D is real-analytic, $\partial D'$ is real algebraic, and let $f : D \to D'$ be a proper holomorphic map. Suppose there exists a point $p \in \partial D$ and a neighbourhood U of p such that f extends smoothly to $\partial D \cap U$. Then the map f extends continuously to $\overline{D}$, and the extension is holomorphic on a dense open subset of ∂D. If D' is strictly pseudoconvex, then f extends holomorphically to a neighbourhood of $\overline{D}$.*

For $n = N$ a similar result is contained in [31]. We note that without the assumption of smooth extension of f *somewhere* on ∂D the conclusion of Theorem 1.2 is false in general. Indeed, there exist proper holomorphic maps of balls of different dimension that do not extend even continuously to the boundary ([18],[12]), or that are continuous up to the boundary but are not of class C^2 ([11],[15]). Further, there exist proper maps $f : \mathbb{B}^n \to \mathbb{B}^N$ which are continuous up to the boundary, and $f(S^{2n-1}) = S^{2N-1}$, provided that N is sufficiently large ([14]).

On the other hand, if f is known to extend smoothly to *all* of ∂D, then f extends holomorphically everywhere on ∂D according to [5] and [22]. We use these results to obtain holomorphic extension of f somewhere on the boundary of D to start analytic continuation along ∂D. Also without the assumption of algebraicity, Forstnerič [13] proved that a proper holomorphic map $f : D \to D'$ between strictly pseudoconvex domains $D \subset \mathbb{C}^n$, $D' \subset \mathbb{C}^N$, $1 < n \le N$, with real analytic boundaries which extends smoothly to ∂D, necessarily extends holomorphically on a dense open subset of ∂D (this was recently improved in [27] by showing that the extension is holomorphic everywhere provided that $1 < n \le N \le 2n$.)

The above stated theorems follow from a more general result asserting a local extension of the map f as a correspondence. More precisely, the following holds.

Theorem 1.3. *Let M (resp. M') be smooth hypersurfaces in $\mathbb{C}^n$ (resp. $\mathbb{C}^N$), $1 < n \le N$, where M is real analytic and minimal, and M' is compact real algebraic. Suppose $\Sigma \subset M$ is a connected open set, and $f : \Sigma \to M'$ is a real analytic CR map. Let $b \in \partial\Sigma$, and $\partial\Sigma \cap M$ be a smooth generic submanifold. Then there exists a neighbourhood $U_b \subset \mathbb{C}^n$ of b such that f extends to a holomorphic correspondence $F : U_b \to \mathbb{C}^N$ with $F(U_b \cap M) \subset M'$.*

We note that in the context of Theorem 1.1 it follows that M is pseudoconvex, however, in Theorem 1.3 neither M nor M' has to be pseudoconvex. The extension given by Theorem 1.3 is guaranteed to be single valued if M' satisfies the property that $Q'_{z'} \cap M' = \{z'\}$ near any $z' \in M'$. In particular this holds if M' is strictly pseudoconvex (cf. [13]).

There are no known results when a similar analytic continuation would hold under the assumption that M' is merely real analytic. The problem is not well understood even in the equidimensional case, where it is only known that the germ of a map $f : M \to M'$ extends along any path on M when both M and M' are strictly pseudoconvex ([24], [34]). The case of different dimensions seems to be even more difficult.

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2. Preliminaries

Let M be a smooth real analytic hypersurface in $\mathbb{C}^n$, $n > 1$, $0 \in M$, and U a neighbourhood of the origin. If U is sufficiently small then $M \cap U$ can be identified by a real analytic defining function $\rho(z, \overline{z})$, and for every point $w \in U$ we can associate to M its so-called Segre variety in U

defined as

$$Q_w = \{z \in U : \rho(z, \overline{w}) = 0\}. \tag{1}$$

Note that Segre varieties depend holomorphically on the variable $\overline{w}$. In fact, in a suitable neighbourhood $U = {}'U \times U_n \subset \mathbb{C}^{n-1} \times \mathbb{C}$ we have

$$Q_w = \left\{z = ('z, z_n) \in U : z_n = h('z, \overline{w})\right\}, \tag{2}$$

where h is a holomorphic function. From the reality condition on the defining function the following basic properties of Segre varieties follow:

$$z \in Q_w \;\Leftrightarrow\; w \in Q_z, \tag{3}$$

$$z \in Q_z \;\Leftrightarrow\; z \in M, \tag{4}$$

$$w \in M \Leftrightarrow \{z \in U : Q_w = Q_z\} \subset M. \tag{5}$$

The set $I_w := \{z \in U : Q_w = Q_z\}$ is itself a complex analytic subset of U. So (5), in particular, implies that if M does not contain non-trivial holomorphic curves, then there are only finitely many points in U that have the same Segre variety (for U sufficiently small). For the proofs of these and other properties of Segre varieties see e.g. [10], [8] or [3].

A hypersurface M is called *minimal* if it does not contain any germs of complex hypersurfaces. In this case the dimension of the set I_w can be positive (but less than $n-1$) for all $w \in M$.

If $f : U \to U'$, $U \subset \mathbb{C}^n$, $U' \subset \mathbb{C}^N$, is a holomorphic map sending a smooth real analytic hypersurface $M \subset U$ into another such hypersurface $M' \subset U'$, and U is as in (2), then $f(z) = z'$ implies $f(Q_z) \subset Q'_{z'}$ for z close to the origin. This invariance property of Segre varieties will play a fundamental role in our arguments. We will also denote by w^s the *symmetric* point of a point $w = ('w, w_n) \in U$, which is by definition the unique point defined by $Q_w \cap \{z \in U : 'z = 'w\}$.

Suppose now that the hypersurface $M \subset \mathbb{C}^N$ is smooth, compact, connected, and defined as the zero locus of a real polynomial $P(z, \overline{z})$. Then we may define Segre varieties associated with M as projective algebraic varieties in $\mathbb{P}^N$. Further, this can be done for every point in $\mathbb{P}^N$. Indeed, let M be given as a connected component of the set defined by

$$\{z \in \mathbb{C}^N : P(z, \overline{z}) = 0\}. \tag{6}$$

We can projectivize the polynomial P to define M in $\mathbb{P}^N$ in homogeneous coordinates

$$\hat{z} = [\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_N], \quad z_k = \frac{\hat{z}_k}{\hat{z}_0}, \quad k = 1, \ldots, N, \tag{7}$$

as a connected component of the set defined by

$$\{\hat{z} \in \mathbb{P}^N : \hat{P}(\hat{z}, \overline{\hat{z}}) = 0\}, \tag{8}$$

where the homogeneous polynomial $\hat{P}$ is defined by

$$\hat{P}(\hat{z}_0, \ldots, \hat{z}_N, \overline{\hat{z}_0}, \ldots, \overline{\hat{z}_N}) = (\hat{z}_0 \overline{\hat{z}_0})^{\deg P} \cdot P\left(\frac{\hat{z}_1}{\hat{z}_0}, \ldots, \frac{\hat{z}_N}{\hat{z}_0}, \frac{\overline{\hat{z}_1}}{\overline{\hat{z}_0}}, \ldots, \frac{\overline{\hat{z}_N}}{\overline{\hat{z}_0}}\right).$$

We may define now the *polar* of M as

$$\hat{M}^c = \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^N \times \mathbb{P}^N : \hat{P}(\hat{z}, \hat{\zeta}) = 0\}. \tag{9}$$

Then $\hat{M}^c$ is a complex algebraic variety in $\mathbb{P}^N \times \mathbb{P}^N$. Given $\tau \in \mathbb{P}^N$, we set

$$\hat{Q}_\tau = \hat{M}^c \cap \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^N \times \mathbb{P}^N : \hat{\zeta} = \overline{\tau}\}. \tag{10}$$

We define the projection of $\hat{Q}_\tau$ to the first coordinate to be the Segre variety of τ.

Recall that for domains $D \subset \mathbb{C}^n$ and $D' \subset \mathbb{C}^N$, a holomorphic correspondence $F : D \to D'$ is a complex analytic set $A \subset D \times D'$ of pure dimension n such that the coordinate projection $\pi : A \to D$ is proper (while $\pi' : A \to D'$ need not be). In this situation, there exists a system of canonical defining functions

$$\Phi_I(z, z') = \sum_{|J| \le m} \Phi_{IJ}(z) z'^J, \ (z, z') \in D \times D', \ |I| = m, \tag{11}$$

where $\Phi_{IJ}(z)$ are holomorphic on D, and A is the set of common zeros of the functions $\Phi_I(z, z')$. For details see, e.g. [4]. It follows that π is in fact surjective and a finite-to-one branched covering. In particular, there exists a complex subvariety $S \subset D$ and a number m such that

$$F := \pi' \circ \pi^{-1} = \{f^1(z), \dots, f^m(z)\}, \tag{12}$$

where f^j are distinct holomorphic maps in a neighborhood of $z \in D \setminus S$. The set S is called the *branch locus* of F. We say that the correspondence F *splits* at $z \in D$ if there is an open subset $U \ni z$ and holomorphic maps $f^j : U \to D'$, $j = 1, 2, \dots, m$, that represent F. Thus F splits at every point $z \in D \setminus S$.

Let M be a smooth real hypersurface $\mathbb{C}^n$. A smooth curve $\gamma : [0, 1] \to M$ is called a *CR-curve*, if for $t \in (0, 1)$, $\gamma'(t) \in H_{\gamma(t)}(M)$, where $H_p(M)$ denotes the complex tangent space to M at a point $p \in M$. We denote by $\mathrm{Orb}(p)$ the set of all points on M which can be connected with p by a piecewise smooth CR curve. $\mathrm{Orb}(p)$ is called the *CR orbit* of p. It is well known that for any $p \in M$, the CR-orbit $\mathrm{Orb}(p)$ is a CR submanifold of M of the same CR dimension. Therefore, if M is minimal, then the CR-orbit of any point $p \in M$ contains an open neighbourhood of p in M. For a detailed discussion of CR-orbits see e.g. [3], or a recent survey [21].

It follows from the above discussion that as a consequence of Theorem 1.1 the map f can be continued analytically to any point on M. In particular, one can say that f extends holomorphically along *any* curve on M.

3. Proof of Theorem 1.3

In the proof of Theorem 1.3 we modify the approach in [30] to our situation. The strategy can be outlined as follows. Without loss of generality we may assume that f is a holomorphic map defined in a neighbourhood of Σ, and $f(\Sigma) \subset M'$. According to [30], Prop. 5.1, there exists a dense open subset ω of Q_b with the property that for $a \in \omega$, $Q_a \cap \Sigma \neq \varnothing$. Furthermore, there exists a non-constant curve $\gamma \subset \Sigma \cap Q_a$ with the endpoint at b. Thus we have a choice of points ξ and a such that

$$a \in Q_b, \quad \xi \in \gamma \subset \Sigma \cap Q_a. \tag{13}$$

The extension of f to the point b can be proved in two steps. Suppose that f is holomorphic in U_ξ, which is some neighbourhood of ξ. Let U be a neighbourhood of Q_ξ. We first show that the set A defined by

$$A = \left\{(w, w') \in U \times \mathbb{C}^N : f(Q_w \cap U_\xi) \subset Q'_{w'}\right\} \tag{14}$$

is complex analytic with the property that A contains Γ_f, the graph of f, and the projection $\pi : A \to U$ is surjective. Further, A can be extended to an analytic subset of $U \times \mathbb{P}^N$, and we denote by $\pi' : A \to \mathbb{P}^N$ the other coordinate projection.

Secondly, we choose suitable neighbourhoods U_a and U^* of a and Q_a respectively, and consider the set

$$A^* = \left\{(w, w') \in U^* \times \mathbb{P}^N : \pi^{-1}(Q_w \cap U_a) \subset \pi'^{-1}(Q'_{w'})\right\}. \tag{15}$$

We then show that A^* also contains the graph of f, and its projection π^* to the first component is also surjective. In particular, $\pi^*(A^*)$ contains a neighbourhood of b. Note that by construction the dimension of A may be bigger than $n = \dim \Gamma_f$. An important fact, however, is that $\dim A^* = n$, regardless of the dimension of the set A. This allows us to show that f extends locally as a holomorphic correspondence to a neighbourhood of b.

3.1. **Extension along Q_ξ.** In this subsection we show that if f is holomorphic at $\xi \in \Sigma$, then we can extend the graph of f as an analytic set along Q_ξ. It follows from (3) that there exist neighbourhoods U_ξ of ξ and U of Q_ξ such that for any point $w \in U$, the set $Q_w \cap U_\xi$ is non-empty. Further, U_ξ and U can be chosen such that $Q_w \cap U_\xi$ is connected for all $w \in U$. We claim that the set defined by (14) is a closed complex analytic subset of $U \times \mathbb{C}^N$. Indeed, the inclusion $f(Q_w \cap U_\xi) \subset Q'_{w'}$ can be expressed (cf. [30]) as

$$P'\left(f('z, h('z, \overline{w})), \overline{w}'\right) = 0, \tag{16}$$

where $P'(z', \overline{z}')$ is the defining polynomial of M', and h is the map defined in (2). After conjugation this becomes a system of holomorphic equations in w and w'. The variable $'z$ plays the role of a parameter here, but from the Noetherian property of the ring of holomorphic functions, we may extract a finite subsystem which defines A as a complex analytic set. Further, since the equations in (16) are polynomials in w', we may projectivize A. This defines an analytic set in $U \times \mathbb{P}^N$, which we denote again by A for simplicity.

Finally, observe that by the invariance property of Segre varieties it follows that A contains the points of the form $(w, f(w))$, $w \in U_\xi$, and therefore A contains the germ at ξ of the graph of f. This also shows that A is not empty. We may consider only the irreducible components of the least dimension which contain Γ_f. Thus we may assume that $\dim A \equiv m \geq n$.

3.2. **Extension along Q_a.** Let $\pi : A \to U$ and $\pi' : A \to \mathbb{P}^N$ be the natural projections. Since $\mathbb{P}^N$ is compact, and A is closed in $U \times \mathbb{P}^N$, the projection π is proper. By the Remmert proper mapping theorem, $\pi(A)$ is a complex analytic subset of U, which simply means that $\pi(A) = U$. For $\zeta \in A$ let $l_\zeta \pi \subset A$ be the germ of the fibre $\pi^{-1}(\pi(\zeta))$ at ζ. Then for a generic point $\zeta \in A$, $\dim l_\zeta \pi = m - n$ which is the smallest possible dimension of the fibre. By the Cartan-Remmert theorem (see e.g. [17]) the set

$$S := \{\zeta \in A : \dim l_\zeta \pi > m - n\} \tag{17}$$

is complex analytic, and by the Remmert proper mapping theorem $\pi(S)$ is complex-analytic in U. We note that $\dim \pi(S) < n - 1$. This can be seen as follows: if $(m - n) + k$ is the generic dimension of the fibre over $\pi(S)$, $k > 0$, then $\dim S = \dim \pi(S) + (m - n + k)$. Since $\dim S \leq m - 1$, $\dim \pi(S) \leq n - 1 - k$, and the assertion holds.

From the above considerations we conclude that $\pi(S)$ does not contain $Q_b \cap U$. The sets U and U_ξ defined in Section 3.1 certainly depend on the choice of ξ. However, if we vary the point ξ in Σ, then the sets defined by (14) with a different choice of ξ will coincide on the overlaps and satisfy the properties stated in Section 3.1. Hence, if $a \in \pi(S) \cap Q_\xi \cap Q_b$, then we may slightly rearrange points $a \in Q_b$ and $\xi \in \Sigma \cap Q_a$, and repeat the above constructions (keeping the same notation), so that $a \notin \pi(S)$.

Let U_a be a neighbourhood of the point a in U, so small that $U_a \cap \pi(S) = \varnothing$. Let $\gamma \subset Q_a \cap \Sigma$ be a path connecting ξ and b. We may choose a neighbourhood U^* of γ (including its endpoints) and U_a in such a way that $Q_w \cap U_a$ is non-empty and connected for any w in U^*. Consider the set A^* defined in (15).

Lemma 3.1. A^* is a complex-analytic subset of $U^* \times \mathbb{P}^N$.

Proof. Let $(w_0, w_0') \in A^*$ be an arbitrary point. Consider $\pi^{-1}(Q_{w_0} \cap U_a)$. This is a complex analytic subset of $A \cap (U_a \times \mathbb{P}^N)$. Since $U_a \cap \pi(S) = \varnothing$, the fibres of π are of constant dimension for points in U_a. Therefore, $\pi^{-1}(Q_{w^0} \cap U_a)$ has constant dimension $m-1$. It follows that analytic sets $\pi^{-1}(Q_w \cap U_a)$ have the same dimension and vary analytically as w varies near w_0. We denote by $B(X, \epsilon)$ the open ϵ-neighbourhood of a set X.

Let $q \in \pi^{-1}(Q_{w_0} \cap U_a)$. Then there exists an affine coordinate patch $U' \subset \mathbb{P}^N$, $q \in U_a \times U'$, with coordinates

$$(z, \zeta') = (z_1, \ldots, z_n, \zeta'_{n+1}, \ldots, \zeta'_{n+N}) \in U_a \times U', \tag{18}$$

and a choice of a coordinate plane in $U_a \times U'$ passing through q, which is spanned by

$$(z_1, z_2, \ldots, z_{n-1}, \zeta'_{k_1}, \zeta'_{k_2} \ldots, \zeta'_{k_{m-n}}) \tag{19}$$

for some $k_1, k_2, \ldots k_{m-n}$, such that for some $\epsilon_q > 0$, the set $\pi^{-1}(Q_{w_0} \cap U_a) \cap B(q, \epsilon_q)$ can be represented as in (11), i.e. as the zero locus of the functions

$$\Phi_I(z, \zeta') = \sum_{\substack{0 \le j \le m_q \\ |J| \le M_q}} \Phi_{IjJ}(z_1, z_2, \ldots, z_{n-1}, \zeta'_{k_1}, \zeta'_{k_2} \ldots, \zeta'_{k_{m-n}})(z_n)^j (\tilde{\zeta}')^J, \ |I| \le l_q, \tag{20}$$

where $\tilde{\zeta}'$ are the remaining $(N-m+n)$ coordinates in U', $J = (j_1, \ldots, j_{N-m+n})$, and Φ_{IjJ} are holomorphic functions. Since $\pi^{-1}(Q_w \cap U_a)$ depend anti-holomorphically on w, there exists $\delta_q > 0$ and a connected open neighbourhood $\Omega_q \subset B(q, \epsilon_q)$ of the point q, such that for $|w - w_0| < \delta_q$ a similar representation also holds for $\pi^{-1}(Q_w \cap U_a) \cap \Omega_q$ with functions

$$\Phi_I(z, \zeta', \overline{w}) = \sum_{\substack{0 \le j \le m_q \\ |J| \le M_q}} \Phi_{IjJ}(z_1, \ldots, z_{n-1}, \zeta'_{k_1}, \ldots, \zeta'_{k_{m-n}}, \overline{w})(z_n)^j (\tilde{\zeta}')^J, \ |I| \le l_q, \tag{21}$$

where the dependence on $\overline{w}$ is holomorphic.

We claim that there exist $\delta > 0$ and a finite collection of points $q^k \in \pi^{-1}(Q_{w_0} \cap U_a)$, $k = 1, 2, \ldots, l$ such that $\cup_{k=1}^{l} \Omega_{q^k}$ has a non-empty intersection with every irreducible component of $\pi^{-1}(Q_w \cap U_a)$, provided that $|w - w_0| < \delta$.

To prove the claim first observe that from compactness of $\mathbb{P}^N$ and continuity of the fibres of the projection π, it follows that given any small $\epsilon > 0$ there exists $\delta > 0$ such that the distance between $\pi^{-1}(Q_w \cap U_a)$ and $\pi^{-1}(Q_{w_0} \cap U_a)$ is less than ϵ whenever $|w - w_0| < \delta$. The distance in $U_a \times \mathbb{P}^N$ can be taken with respect to the product metric of the standard metric in $\mathbb{C}^n$ and the Fubini-Study metric in $\mathbb{P}^N$.

Denote by S_w^j the irreducible components of $\pi^{-1}(Q_w \cap U_a)$, $j = 1, \ldots, l_w$, where w is a point in some small neighbourhood of w_0. Choose $\epsilon_1 > 0$ and $\delta_1 > 0$ such that for $|w - w_0| < \delta_1$, none of the components S_w^j is entirely contained in $B(\partial U_a \times \mathbb{P}^N, \epsilon_1)$. Such ϵ_1 and δ_1 exist because every S_w^j surjectively projects onto $Q_w \cap U_a$. Then $(U_a \times \mathbb{P}^N) \setminus B(\partial U_a \times \mathbb{P}^N, \epsilon_1)$ is compact, and therefore, the open cover of the set

$$\pi^{-1}(Q_{w_0} \cap U_a) \setminus B(\partial U_a \times \mathbb{P}^N, \epsilon_1) \tag{22}$$

by Ω_q, where $q \in \pi^{-1}(Q_{w_0} \cap U_a)$, admits a finite subcover, say, $\Omega_{q^1}, \ldots \Omega_{q^l}$. Let

$$\epsilon_2 = \min_{k=1,\ldots,l} \left\{ \sup\{\alpha > 0 : B(q^k, \alpha) \subset \Omega_{q^k}\} \right\}. \tag{23}$$

Then there exists δ_2 such that the distance between $\pi^{-1}(Q_w \cap U_a)$ and $\pi^{-1}(Q_{w_0} \cap U_a)$ is less than ϵ_2 whenever $|w - w_0| < \delta_2$. Finally, choose $\delta = \min\{\delta_1, \delta_2\}$. Then for any w with $|w - w_0| < \delta$, any component S_w^j has a non-empty intersection with $\cup_k \Omega_{q^k}$. This proves the claim.

We now show that A^* is complex-analytic in a neighbourhood of a point $(w_0, w_0') \in A^*$. Choose $q^1, \dots q^l$ as claimed above. We fix some q^k, $k \in \{1, 2, \dots, l\}$ and let $\eta = \overline{w}$, $\eta' = \overline{w}'$. Let further $G = \Omega_{q^k} \times \{|(\eta, \eta') - (\eta_0, \eta_0')| < \delta\}$ be a small neighbourhood of $(q^k, \overline{w}_0, \overline{w}_0')$ in $\mathbb{C}^n_z \times \mathbb{C}^N_{\zeta'} \times \mathbb{C}^n_\eta \times \mathbb{C}^N_{\eta'}$. We define

$$X_1 = \{(z, \zeta', \eta, \eta') \in G : P'(\zeta', \eta') = 0\}, \tag{24}$$

$$X_2 = \{(z, \zeta', \eta, \eta') \in G : \Phi_I^k(z, \zeta', \eta) = 0, \ |I| \le l_{q^k}\}, \tag{25}$$

where $\Phi_I^k(z, \zeta', \eta)$ are holomorphic functions in as defined in (21). Both of these sets are complex analytic in G. Then the set of points (w, w') for which the inclusion

$$\pi^{-1}(Q_w \cap U_a) \cap \Omega_{q^k} \subset \pi'^{-1}(Q'_{w'}) \tag{26}$$

holds is conjugate to the set X^* in the (η, η') space which is characterized by the property that $(\eta, \eta') \in X^*$ whenever $\pi_2^{-1}(\eta, \eta') \subset \pi_1^{-1}(\eta, \eta')$, where π_j is the coordinate projection from X_j to the (η, η')-space. The set X^* can be also defined as

$$X^* = \{(\eta, \eta') : \dim \pi_2^{-1}(\eta, \eta') = \dim \pi_{12}^{-1}(\eta, \eta')\}, \tag{27}$$

where $\pi_{12} : X_1 \cap X_2 \to \mathbb{C}^{n+N}_{(\eta, \eta')}$. Further, $\dim \pi_2^{-1}(\eta, \eta') = m-1$, for all (η, η'), and so $\dim \pi_{12}^{-1}(\eta, \eta') \le m - 1$. Thus, $X^* = \pi_{12}(\tilde{X})$, where

$$\tilde{X} = \{(z, \zeta', \eta, \eta') \in X_1 \cap X_2 : \dim l_{(z, \zeta', \eta, \eta')} \pi_{12} > m - 2\}. \tag{28}$$

By the Cartan-Remmert theorem $\tilde{X}$ is a complex analytic subset of G. Denote by $\tilde{\pi}$ the projection from $\tilde{X}$ to the space of variables $(z_1, \dots, z_{n-1}, \zeta_1, \dots, \zeta_{k_{m-n}}, \eta, \eta')$. By construction of functions in (21) the map $\tilde{\pi}$ is proper. Hence, by the Remmert proper mapping theorem, $\tilde{\pi}(\tilde{X})$ is complex analytic. Finally, consider the projection $\pi_{(\eta, \eta')} : \tilde{\pi}(\tilde{X}) \to (\eta, \eta')$. From the construction of the set $\tilde{\pi}(\tilde{X})$, $\dim \pi^{-1}_{(\eta, \eta')}(\eta, \eta') = m - 1$, for $(\eta, \eta') \in X^*$. But in fact, $\dim \pi^{-1}_{(\eta, \eta')}(\pi_{(\eta, \eta')}(x)) = m - 1$, for any $x \in \tilde{\pi}(\tilde{X})$. Thus we may identify X^* with $\tilde{\pi}(\tilde{X}) \cap \{(z_1, \dots, z_{n-1}, \zeta_1, \dots, \zeta_{k_{m-n}}) = \text{const}\}$. This proves that the set X^* is complex analytic. After conjugation, we may assume that the set defining the inclusion in (26) is also complex analytic.

If an open set of the irreducible component S_w^j is contained in $\pi'^{-1}(Q'_{w'})$ for some w', then by the uniqueness theorem, the whole component S_w^j must be contained in $\pi'^{-1}(Q'_{w'})$. Therefore, since $\cup_{k=1}^l \Omega_{q^k}$ has a non-empty intersection with every S_w^j, the system of equations defining the inclusion (26), combined for $k = 1, \dots, l$, completely determines the inclusion in (15), and therefore it defines A^* as a complex-analytic set near (w_0, w_0').

So far we have showed that A^* is a *local* complex analytic set, i.e defined by a system of holomorphic equations in a neighbourhood of any of its points. To prove that A^* is a complex-analytic *subset* of $U^* \times \mathbb{P}^N$ it is enough now to show that A^* is closed in $U^* \times \mathbb{P}^N$. Suppose $(w^j, w'^j) \to (w^0, w'^0)$, as $j \to \infty$, for some sequence $(w^j, w'^j) \in A^*$, and suppose that $(w^0, w'^0) \in U_a \times \mathbb{P}^N$. This means that $\pi^{-1}(Q_{w^j} \cap U_a) \subset \pi'^{-1}(Q'_{w'^j})$. Since $Q_{w^j} \to Q_{w^0}$ and $Q'_{w'^j} \to Q'_{w'^0}$, by analyticity also $\pi^{-1}(Q_{w^0} \cap U_a) \subset \pi'^{-1}(Q'_{w'^0})$, and therefore $(w^0, w'^0) \in A^*$. This completes the proof of Lemma 3.1. □

Lemma 3.2. The set A^* contains the germ of the graph of f at $(\xi, f(\xi))$. Further,

$$A^* \cap \big((U_\xi \cap U \cap U^*) \times \mathbb{P}^N\big) \subset A.$$

Proof. Suppose $z \in (U_\xi \cap U \cap U^*)$. We need to show that

$$\pi^{-1}(Q_z \cap U_a) \subset \pi'^{-1}\left(Q'_{f(z)}\right). \tag{29}$$

Let $w \in Q_z \cap U_a$ be an arbitrary point, and let $(w, w') \in A$. Then $f(Q_w \cap U_\xi) \subset Q'_{w'}$. In particular, since $z \in Q_w \cap U_\xi$, we have $f(z) \in Q'_{w'}$. But this implies $w' \in Q'_{f(z)}$. In other words, $(w, w') \in \{w\} \times Q'_{f(z)}$. Since w' was an arbitrary point in A over w, we conclude that $\pi^{-1}(w) \subset \pi'^{-1}\left(Q'_{f(z)}\right)$. Consequently, (29) follows, and $(z, f(z)) \in A^*$.

As for the second assertion, we observe that for $(w, w') \in A^*$, where w is sufficiently close to ξ, the inclusion $\pi^{-1}(Q_w \cap U_a) \subset \pi'^{-1}(Q'_{w'})$ is equivalent to $\pi^{-1}(Q_w \cap U_\xi) \subset \pi'^{-1}(Q'_{w'})$, because $Q_w \cap U$ is connected. From Section 3.1 the set A contains the germ of the graph of f near ξ, and therefore, the inclusion $\Gamma_f \subset A^*$ in particular implies $f(Q_w \cap U_\xi) \subset Q'_{w'}$, which by definition means $(w, w') \in A$. ☐

Lemma 3.2 shows that A^* is non-empty. Also note that since $\mathbb{P}^N$ is compact, the projection $\pi^* : A^* \to U^*$ is proper, and therefore, $\pi^*(A^*) = U^*$. Define $\pi'^* : A^* \to \mathbb{P}^N$.

3.3. **Extension as a correspondence.** Let now Ω be a small connected neighbourhood of the path $\gamma \subset Q_a \cap M$, which connects ξ and b, such that for any $w \in \Omega$, the symmetric point w^s belongs to U^*, and let Q^s_w denote the connected component of $Q_w \cap U^*$ which contains w^s. Denote further by S^* the set of points $z \in U^*$ for which $\pi^{*-1}(z) \subset A^*$ does not have the generic dimension. The same argument as at the beginning of Section 3.2 shows that S^* is a complex analytic set of dimension at most $n-2$, and so $\Omega \setminus S^*$ is connected. To prove the extension of f to the point b we will need the following result.

Lemma 3.3. For any point $w \in \Omega \setminus S^*$,

$$\pi^{*-1}(Q^s_w) \subset \pi'^{*-1}(Q'_{w'}), \quad \forall\, w' \in \pi'^* \circ \pi^{*-1}(w). \tag{30}$$

Proof. Denote by Z the set of points in $\Omega \setminus S^*$ for which (30) holds. We show that $Z = \Omega \setminus S^*$. For the proof we shrink U_ξ so that $U_\xi \subset \Omega$.

Let $w \in U_\xi \setminus S^*$ be some point, and $(w, w') \in A^*$. Note that $z^s = z$ for any $z \in M$, and therefore, for w sufficiently close to ξ, the set $Q_w \cap U_\xi$ coincides with $Q^s_w \cap U_\xi$. Let $z \in Q_w \cap U_\xi$ be arbitrary. Then $(z, z') \in A^*$ means $\pi^{-1}(Q_z \cap U_a) \subset \pi'^{-1}(Q'_{z'})$. For z and w sufficiently close to ξ, Q_z is connected in U, and therefore, $\pi^{-1}(Q_z \cap U_\xi) \subset \pi'^{-1}(Q'_{z'})$. The last inclusion in particular means that $\pi^{-1}(w) \subset \pi'^{-1}(Q'_{z'})$. Thus for any $w' \in \pi' \circ \pi^{-1}(w)$, $w' \in Q'_{z'}$, or $z' \in Q'_{w'}$. By Lemma 3.2, A^* is contained in A near ξ, and it follows that for any $w' \in \pi^{*\prime} \circ \pi^{*-1}(w)$, $z' \in Q'_{w'}$. From that (30) follows, and we proved that the set Z contains a small neighbourhood of ξ.

Let Z° be the largest connected open set which contains ξ and is contained in Z. From the above considerations, $Z^\circ \neq \varnothing$. We show that if $w \in (\overline{Z^\circ} \setminus Z^\circ) \cap (\Omega \setminus S^*)$, then $w \in Z^\circ$. Let $(w, w') \in A^*$ for some w'. Since $\dim S^* < \dim Q^s_w = n-1$, we may find a point $\alpha \in (Q^s_w \setminus S^*)$, and by repeating the argument of Lemma 3.1 we may construct a complex analytic set

$$A_w = \{(x, x') \in U_w \times \mathbb{P}^N : \pi^{*-1}(Q^s_x \cap U_\alpha) \subset \pi^{*\prime -1}(Q'_{x'})\}, \tag{31}$$

where U_w and U_α are suitably chosen neighbourhoods of w and α respectively. For every point $x \in U_w \cap Z^\circ$, and every x' such that $(x, x') \in A^*$, the inclusion in (31) holds. This implies

$$A^* \cap \big((Z^\circ \cap U_w) \times \mathbb{P}^N\big) \subset A_w, \tag{32}$$

and in particular, A_w is non-empty. By the uniqueness theorem, it follows that $A^* \cap (U_w \times \mathbb{P}^N) \subset A_w$, and therefore, the projection from A_w to the first component is surjective. Thus, for any $x \in U_w$, the set $Q_x \cap U_\alpha$ (and therefore Q^s_x) will be "mapped" by A^* into Segre variety of a point x', whenever $(x, x') \in A^*$. Hence, $U_w \subset Z^\circ$.

Since $\Omega \setminus S^*$ is connected, it follows now that $Z = \Omega \setminus S^*$. □

We now consider only an irreducible component of A^* which has the smallest dimension, and such that it contains the germ of the graph of f at ξ. Denote for simplicity this component again by A^*. Note that Lemma 3.3 still holds for the new A^*.

Lemma 3.4. $\dim A^* = n$.

Proof. Since $\pi^* : A^* \to U^*$ is surjective, for any $z \in M \cap U^*$ the set ${\pi^*}^{-1}(z)$ is non-empty. We show that for a given $z_0 \in \Sigma \setminus S^*$, the set ${\pi^*}^{-1}(z_0)$ is discrete near $(z_0, f(z_0)) \in A^*$. Indeed, by Lemma 3.3, $(z, z') \in A^* \setminus {\pi^*}^{-1}(S^*)$ implies ${\pi^*}^{-1}(Q^s_z) \subset {\pi'^*}^{-1}(Q'_{z'})$. In particular this means that $\pi'^*({\pi^*}^{-1}(z)) \subset Q'_{z'}$, which implies that $z' \in Q'_{z'}$. Then from (4) it follows that for any $z \in M$ close to z_0, and any z' close to $f(z_0)$, the inclusion $(z, z') \in A^*$ implies $z' \in M'$. Since $\pi'^*({\pi^*}^{-1}(z))$ is a locally countable union of complex analytic sets, and M' contains no non-trivial germs of complex analytic varieties by [7], it follows that ${\pi^*}^{-1}(z)$ is discrete near $(z_0, f(z_0))$. This means that $\dim A^* = n$ near $(z_0, f(z_0))$. But then the lemma follows, since $\dim A^*$ is constant. □

To finish the proof of the theorem, we consider two cases. First, suppose that $b \notin S^*$. Since M' is compact, the cluster set of $f|_\gamma(b)$ is well-defined. Let b' be a point in the cluster set of the point b. It is enough to show now that there exist neighbourhoods $U_b \ni b$ and $U'_{b'} \ni b'$ such that $A^* \cap U_b \times U'_{b'}$ is a holomorphic correspondence. By construction, $(b, b') \in A^*$, and from the proof of Lemma 3.4 we conclude that ${\pi^*}^{-1}(b)$ is discrete near (b, b'). Therefore we may choose U_b and $U'_{b'}$ in such a way that $A^* \cap (U_b \times \partial U'_{b'}) = \varnothing$. It follows then that $\pi^*|_{A^* \cap (U_b \times U'_{b'})}$ is a proper map, and therefore,

$$F := \pi'^*|_{A^* \cap (U_b \times U'_{b'})} \circ {\pi^*}^{-1}|_{U_b} \tag{33}$$

is the desired extension of f as a holomorphic correspondence.

Secondly, suppose $b \in S^*$. Consider a sequence of points $w^j \in (\Sigma \cap \Omega) \setminus S^*$ such that $w^j \to b$ and $\lim f(w^j) = b'$ for some $b' \in M'$. Then

$${\pi^*}^{-1}(Q^s_{w^j}) \subset {\pi'^*}^{-1}(Q'_{f(w^j)}).$$

It follows that

$${\pi^*}^{-1}(Q^s_b) \subset Q'_{b'}. \tag{34}$$

Indeed, it is enough to prove this inclusion in a neighbourhood of any point in Q^s_b. Since $\dim S^* < \dim Q_b$, we may choose this point to be outside S^*. The inclusion then follows by analyticity of the fibres of $\pi^* : A^* \to U^*$.

As in the proof of Lemma 3.4, it follows from (34) that ${\pi^*}^{-1}(b)$ is discrete near (b, b'), and the same argument as above shows that f extends to a neighbourhood of b as a holomorphic correspondence.

Finally, if F is the extension of f as a correspondence, then $F(M) \subset M'$. The reason again is that if $z \in M$ and $z' \in F(z)$, then $F(Q_z) \subset Q'_{z'}$ by (30) and (34), which implies $z' \in Q'_{z'}$, and by (4), $z' \in M'$.

This completes the proof of Theorem 1.3.

4. Proof of other results.

4.1. **Proof of Theorem 1.1.** We first show that the map f can be extended holomorphically along any smooth CR-curve γ on M, i.e. for which the tangent vector to γ at any point is contained in the complex tangent to M. For this we use the construction of a family of ellipsoids as it is done in [20]. We refer to this paper for details of the construction. Let q be the first point on γ to which f does not extend holomorphically. Near q there exists a smooth CR vector field L such that γ is contained in an integral curve of L. By integrating L we obtain a smooth coordinate system $(t, s) \in \mathbb{R} \times \mathbb{R}^{2n-2}$ on M such that for any fixed s_0 the segments (t, s_0) are contained in the trajectories of L. We may further choose a point $p \in \gamma$ sufficiently close to q, so that f is holomorphic near p. After a translation, assume that $p = (0, 0)$. For $\epsilon > 0$ define the family of ellipsoids on M by

$$E_\tau = \{(t, s) : |t|^2/\tau + |s|^2 < \epsilon\}, \tag{35}$$

where $\epsilon > 0$ is so small that for some $\tau_0 > 0$ the ellipsoid E_{τ_0} is compactly contained in the portion of M where f is holomorphic. Then ∂E_τ is generic at every point except the set

$$\Lambda = \{(0, s) : |s|^2 = \epsilon\}.$$

Let further $\tau_1 > 0$ be such that $q \in \partial E_{\tau_1}$.

To prove that f extends holomorphically to a neighbourhood of q we argue by contradiction. For that we assume that τ^* is the smallest positive number such that f does not extend holomorphically to *some* point on ∂E_{τ^*}, and assume that $\tau^* < \tau_1$. By construction, $\tau^* > \tau_0$. Also by construction, near any point $b \in \partial E_{\tau^*}$ to which f does not extend holomorphically, the set ∂E_{τ^*} is a smooth generic submanifold of M, since the non-generic points of ∂E_{τ^*} are contained in Λ, where f is already known to be holomorphic. Then by Theorem 1.3 the map f extends as a correspondence F to a neighbourhood of b.

We now show that F is single valued. Suppose $w' \in F(w)$ for $w \in M$, then by the invariance of Segre varieties $F(Q_w) \subset Q'_{w'}$, and in particular, $w' \in Q'_{w'}$. But since M' is strictly pseudoconvex, in a sufficiently small neighbourhood of $w' \in M'$ there exists only one point on M' whose Segre variety contains w', namely $Q'_{w'}$ itself. Thus the correspondence F splits into several holomorphic maps, one of which by analyticity extends the map f.

This shows that τ^* cannot be smaller than τ_1, which proves that the map f extends holomorphically to q, and therefore along any CR-curve on M.

Finally, observe that minimality of M implies that CR-orbit of any point on M coincides with M. Therefore, using analytic continuation along CR-curves we obtain continuation of f to a neighbourhood of any point on M.

4.2. **Proof of Theorem 1.2.** By [5] and [22] it follows that smooth extension of f implies holomorphic extension to a neighbourhood of p. Let $\omega \subset M$ be a open set where the extension of f is holomorphic, and suppose $b \in \partial\omega$ is a point near which $\partial\omega$ is a smooth generic manifold. Then by Theorem 1.3 the map f admits an extension as a holomorphic correspondence $F : U_b \to \mathbb{C}^N$, where U_b is some neighbourhood of b. This, in particular, proves that f extends continuously to $U_b \cap \partial D$. Indeed, if $q \in U_b \cap \partial D$, then the cluster set of q with respect to f must be contained in the set $F(q)$ which is finite. Since the cluster set is connected it must reduce to a single point thereby showing that f is continuous at q.

Further, by the splitting property of correspondences, there exists a complex analytic subset $S \subset U_b$ such that near any point $q \in U_b \setminus S$, F can be represented by a finite collection of holomorphic mappings. It follows from the uniqueness theorem, that for $q \in \partial D \setminus S$ one of the maps of the splitting must coincide with the map f defined in D. This proves holomorphic extension of f to a dense open subset of $\partial D \cap U_b$.

Let Σ be the largest subset of ∂D defined by the property that if $a \in \Sigma$, then there is a neighbourhood $U_a \ni a$ in $\mathbb{C}^n$ such that f extends continuously to $\partial D \cap U_a$ and holomorphically past a dense open subset of $\partial D \cap U_a$. Then Σ is relatively open in ∂D by definition and non-empty by assumption. We show that Σ does not contain any boundary points. Suppose on the contrary that $q \in \partial\Sigma = \overline{\Sigma} \setminus \Sigma$. Let γ be a CR-curve passing through q and entering Σ. Such γ exists by the minimality of M. Let $p \in \gamma \cap \Sigma$ be close to q. We now repeat the construction of the family of ellipsoids used in the proof of Theorem 1.1. Let E_τ be defined by (35) and centered at p. Then for some $\tau_0 > 0$, E_{τ_0} touches $\partial\Sigma$ at some point, say b (which may be different from q). We now claim that $b \in \Sigma$. This will yield a contradiction to the assumption that Σ has a nonempty boundary, thus proving the first half of the theorem.

As in the proof of Theorem 1.3, since $b \in \partial E_{\tau_0}$ is a generic point, there exists a dense open set $\omega \in Q_b$ such that for $a \in \omega$, $Q_a \cap E_{\tau_0}$ contains a curve with the end point at b. Fix some $a \in \omega$, and suppose that

$$Q_a \cap E_{\tau_0} \not\subset S, \tag{36}$$

where S is the branching locus of the correspondence F extending f. Then we choose a point $\zeta \in (Q_a \cap E_{\tau_0}) \setminus S$, and a branch of F which gives holomorphic extension of f near ζ. We now may repeat the proof of Theorem 1.3 to show that f extends as a correspondence to a neighbourhood of b. If $Q_a \cap E_{\tau_0} \subset S$, we simply choose another point $a \in \omega$ so that (36) holds. This proves that f extends as a correspondence to a neighbourhood of b, and thus $b \in \Sigma$.

Finally, the second statement of the theorem follows immediately from Theorem 1.1.

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