DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **12**, Number **3**, March **2005**

Website: http://aimSciences.org

pp. **403–412**

STABLE SETS, HYPERBOLICITY AND DIMENSION

Rasul Shafikov and Christian Wolf

Department of Mathematics
Middlesex College
The University of Western Ontario
London, Ontario N6A 5B7, Canada

Department of Mathematics
Wichita State University
Wichita, Kansas, 67260, USA

(Communicated by Marcelo Vina)

Abstract. In this note we derive an upper bound for the Hausdorff and box dimension of the stable and local stable set of a hyperbolic set Λ of a C^2 diffeomorphisms on a n-dimensional manifold. As a consequence we obtain that $\dim_H W^s(\Lambda) = n$ is equivalent to the existence of a SRB-measure. We also discuss related results for expanding maps.

1. **Introduction.** Let M be an n-dimensional smooth Riemannian manifold, $f : M \to M$ a diffeomorphism, and $\Lambda \subset M$ a locally maximal hyperbolic set of f. Given $\varepsilon > 0$, we define the local stable set of Λ by

$$W^s_\varepsilon(\Lambda) = \{x \in M : \mathrm{dist}(f^k(x), \Lambda) < \varepsilon \text{ for all } k \in \mathbb{N}\}. \tag{1}$$

In this paper we investigate the complexity of $W^s_\varepsilon(\Lambda)$ in terms of its upper box dimension $\overline{\dim}_B W^s_\varepsilon(\Lambda)$. It is a classical result of Bowen [3] that there exist examples of C^1 horseshoes Λ with positive Lebesgue measure, in particular $\overline{\dim}_B W^s_\varepsilon(\Lambda) = n$. On the other hand, by a result of Bowen and Ruelle [5], if f is a C^2-diffeomorphism and Λ is not an attractor then $W^s_\varepsilon(\Lambda)$ has zero Lebesgue measure. We extend the latter result by showing that the upper box dimension of $W^s_\varepsilon(\Lambda)$ is strictly smaller than n. More precisely, we derive an upper bound for the upper box dimension of $W^s_\varepsilon(\Lambda)$ which is given in terms of the exponential expansion rate of the tangent map defined by

$$s = \lim_{k\to\infty} \frac{1}{k} \log\left(\max\{||Df^k(x)|| : x \in \Lambda\}\right), \tag{2}$$

and the topological pressure of the unstable Jacobian. If Λ is not an attractor this bound is strictly smaller than n. Our main result is the following:

2000 *Mathematics Subject Classification.* 37C45, 37C40, 37DXX.
Key words and phrases. hyperbolic sets, SRB measure, Hausdorff dimension, box dimension, stable sets.
The research of C.W. was supported in part by the National Science Foundation under Grant No. EPS-0236913 and matching support from the State of Kansas through Kansas Technology Enterprise Corporation.

Theorem 1. *Let $f : M \to M$ be a C^2-diffeomorphism, and Λ a locally maximal hyperbolic set of f, which is not a periodic orbit, such that $f|\Lambda$ is topologically mixing. Define $\phi^u = -\log|\det Df|E^u|$. Then*

$$\overline{\dim}_B W^s_\varepsilon(\Lambda) \leq n + \frac{P(\phi^u)}{s}. \tag{3}$$

Here $P(\phi^u)$ denotes the topological pressure of ϕ^u, see Section 2 for details. We note that our result holds for manifolds of arbitrary dimension, in particular, we do not require f to be conformal on Λ. Since f is a diffeomorphism, the analogous result also holds for the local unstable set $W^u_\varepsilon(\Lambda)$ of Λ. We define the stable set of Λ by

$$W^s(\Lambda) = \{x \in M : \mathrm{dist}(f^k(x), \Lambda) \to 0 \text{ for } k \to \infty\}. \tag{4}$$

As a consequence of Theorem 1 we obtain that the upper bound in (3) also provides an upper bound for the Hausdorff dimension of $W^s(\Lambda)$ (see Corollary 7).

Another consequence of Theorem 1 is the following:

Corollary 2. *Suppose that f and Λ are as in Theorem 1, and assume that Λ has empty interior. Then $\overline{\dim}_B\Lambda$ is strictly smaller than n.*

We note that the case when Λ has non-empty interior actually occurs. For example, if f is an Anosov diffeomorphism, then the entire manifold M is a hyperbolic set. The result of Corollary 2 was known in some special cases. In particular, if M is a surface, that is, $n = 2$, then the classical result of McCluskey and Manning [12] states that the Hausdorff dimension of Λ coincides with the box dimension of Λ, and that its value is strictly smaller than 2. Recently, significant progress has also been made toward the estimation of the dimension of hyperbolic sets of higher dimensional manifolds. In [1] Barreira applied the non-additive topological pressure to derive estimates for the unstable/stable slice dimensions (i.e. the dimension of the intersection of the unstable/stable manifolds with the hyperbolic set). Later, Franz [6] obtained upper bounds for the dimension of an invariant set admitting an equivariant splitting in terms of the topological entropy and the uniform Lyapunov exponents. Her results particularly apply to hyperbolic sets of diffeomorphisms. Further, Barreira [2] gave estimates for the dimension of a repeller of a expanding map by using the non-additive topological pressure and singular value functions. We refer to [13] for related results.

All these results provide upper bounds for the dimension of a hyperbolic set in terms of other important dynamically defined quantities. However, it is often difficult to calculate these upper bounds for concrete examples. Corollary 2, on the other hand, states without further calculations that the dimension of a hyperbolic set is strictly smaller than n. This has not been known before in this generality.

Another consequence of Theorem 1 is a new characterization for f to have an invariant probability measure μ supported on Λ whose conditional measures on the unstable manifolds are absolutely continuous with respect to the Lebesgue measure. Such a measure μ is called an SRB measure of the diffeomorphism f.

Corollary 3. *Suppose that f and Λ are as in Theorem 1, and let $\varepsilon > 0$ be small. Then the following are equivalent.*

(i) $\overline{\dim}_B W^s_\varepsilon(\Lambda) = n$;
(ii) $\dim_H W^s(\Lambda) = n$;
(iii) *f admits a SRB measure on Λ;*
(iv) *Λ is an attractor of f;*

(v) $W^s(\Lambda)$ *has positive Lebesgue measure.*

The novelty of Corollary 3 is the implication (ii) implies (iii), while the other implications are well-known, and are not consequences of our results.

This paper is organized as follows. In Section 2 we consider hyperbolic diffeomorphisms and derive an upper bound of the box dimension of the local stable set of a hyperbolic set. Furthermore, we apply this bound to establish the corollaries stated in the introduction. Finally, in Section 3 we study repellers of expanding maps, and derive related results as in the case of diffeomorphisms.

2. **Stable sets for diffeomorphisms.** Let $f\colon M \to M$ be a C^2 diffeomorphism on an n-dimensional Riemannian manifold, and $\Lambda \subset M$ a hyperbolic set. This means that Λ is a compact f-invariant set, and that there exist a continuous splitting of the tangent bundle $T_\Lambda M = E^u \oplus E^s$, and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

1. $Df(x)(E^u_x) = E^u_{f(x)}$ and $Df(x)(E^s_x) = E^s_{f(x)}$;
2. $\|Df^{-k}(x)v\| \le c\lambda^k \|v\|$ whenever $v \in E^u_x$ and $k \in \mathbb{N}$;
3. $\|Df^{k}(x)v\| \le c\lambda^k \|v\|$ whenever $v \in E^s_x$ and $k \in \mathbb{N}$.

We say that Λ is locally maximal if there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{k\in\mathbb{Z}} f^k U$. We shall always assume that Λ is a locally maximal hyperbolic set, which is not a periodic orbit, and that $f|\Lambda$ is topologically mixing.

We say that Λ is an attractor of f if there are arbitrarily small neighborhoods U of Λ such that $f(U) \subset U$. This includes the case of Anosov diffeomorphisms for which the entire manifold M is a hyperbolic set. We say that Λ is a repeller of f if it is an attractor of f^{-1}. We note that there are other notions of an attractor and repeller. Here we follow the definitions of [4] which suit well our purposes.

Given $k \in \mathbb{N}$, we define a metric d_k on M given by

$$d_k(x,y) = \max_{i=0,\ldots,k-1} d(f^i(x), f^i(y)), \tag{5}$$

where $d(.,.)$ denotes the distance induced by the Riemannian metric on M. Following Bowen [4], for $x \in \Lambda$, $\varepsilon > 0$ and $k \in \mathbb{N}$ we define the sets

$$B(x,\varepsilon,k) = \{y \in M : d_k(x,y) < \varepsilon\} \tag{6}$$

and

$$B(\Lambda,\varepsilon,k) = \bigcup_{x\in\Lambda} B(x,\varepsilon,k). \tag{7}$$

Recall that $W^s_\varepsilon(\Lambda)$ denotes the local stable set of Λ (see (1)). It is an immediate consequence of the shadowing lemma that if ε is sufficiently small, then there exists $\overline{\varepsilon} > 0$ such that

$$W^s_\varepsilon(\Lambda) \subset \bigcup_{x\in\Lambda} W^s_{\overline{\varepsilon}}(x), \tag{8}$$

where $W^s_{\overline{\varepsilon}}(x)$ denotes the local stable manifold of size $\overline{\varepsilon}$ of $x \in \Lambda$. Furthermore, by choice of ε, the number $\overline{\varepsilon}$ can be chosen arbitrarily small.

We now define the topological pressure (see for example [16] for a detailed discussion). Let $\delta > 0$. A set $E \subset M$ is called (k,δ)-separated with respect to f if $d_k(x,y) < \delta$ implies $x = y$ for all $x, y \in E$. For all $(k,\delta) \in \mathbb{N} \times \mathbb{R}^+$ let $E_k(\delta)$ be a maximal (k,δ)-separated subset of Λ with respect to f (in the sense of inclusion).

We denote by $C(\Lambda,\mathbb{R})$ the Banach space of all continuous functions from Λ to $\mathbb{R}$. The topological pressure of f is a mapping $P: C(\Lambda,\mathbb{R})\to\mathbb{R}$ defined by

$$P(\varphi)=\lim_{\delta\to 0}\limsup_{k\to\infty}\frac{1}{k}\log\left(\sum_{x\in E_k(\delta)}\exp\left(\sum_{i=0}^{k-1}\varphi\circ f^i(x)\right)\right). \tag{9}$$

We are particularly interested in the topological pressure of the function $\phi^u:\Lambda\to\mathbb{R}$ defined by $\phi^u(x)=-\log|\det Df(x)|E_x^u|$, where $\det Df(x)|E_x^u$ denotes the Jacobian of the linear map $Df(x)|E_x^u$. We will need the following result which is due to Bowen.

Proposition 4 ([4]). *If $\varepsilon>0$ is small enough then:*

(i) $\lim_{k\to\infty}\frac{1}{k}\log(\mathrm{vol}(B(\Lambda,\varepsilon,k))=P(\phi^u)\le 0$;
(ii) $P(\phi^u)=0$ *if and only if* $W_\varepsilon^s(\Lambda)$ *has nonempty interior, in which case* Λ *is an attractor.*

We note that the right-hand side inequality in (i) also follows as an application of the Margulis-Ruelle inequality and the variational principle.

We recall the definitions of some fractal dimensions that we use in the paper (see e.g. [11] for details). Let $A\subset M$ and $s\ge 0$. The s-dimensional Hausdorff measure of A is defined by

$$H^s(A)=\sup_{\varepsilon>0}\inf\left\{\sum_{k=1}^{\infty}\mathrm{diam}(U_k)^s: A\subset\bigcup_{k=1}^{\infty}U_k,\ \mathrm{diam}(U_k)\le\varepsilon\right\},$$

where $\mathrm{diam}(U_k)$ denotes the diameter of the set U_k with respect to the Riemannian metric on M. We define the Hausdorff dimension of A by

$$\dim_H A=\inf\{s: H^s(A)=0\}=\sup\{s: H^s(A)=\infty\}.$$

For a relatively compact set A we denote by $N_\varepsilon(A)$ the least number of balls with radius ε needed to cover A. We define the lower and upper box dimensions of A by

$$\underline{\dim}_B A=\liminf_{\varepsilon\to 0}\frac{\log N_\varepsilon(A)}{-\log\varepsilon}\quad\text{and}\quad\overline{\dim}_B A=\limsup_{\varepsilon\to 0}\frac{\log N_\varepsilon(A)}{-\log\varepsilon}.$$

It follows that

$$\dim_H A\le\underline{\dim}_B A\le\overline{\dim}_B A, \tag{10}$$

where equality holds for sufficiently regular sets (see [11] for details).

We now prove Theorem 1 stated in the introduction.

Proof of Theorem 1. We first remark that it follows from Proposition 4 that if Λ is not an attractor, then $P(\phi^u)<0$, and therefore inequality (3) provides a non-trivial estimate.

Observe that since the operator norm is submultiplicative, the limit defining s exists (see e.g. [16]). Furthermore, since Λ is not an attracting cycle, $s>0$. If Λ is an attractor, then by Proposition 4, $P(\phi^u)=0$, and inequality (3) trivially holds. Thus, we may assume that Λ is not an attractor, in which case $P(\phi^u)<0$.

Let $\delta>0$. It follows from a simple continuity argument that there exist $\varepsilon>0$ and $k_\delta\in\mathbb{N}$ such that for all $x\in B(W_\varepsilon^s(\Lambda),\varepsilon)=\{x\in M:\exists y\in W_\varepsilon^s(\Lambda),\ d(x,y)<\varepsilon\}$ we have

$$\|Df^{k_\delta}(x)\|<\exp(k_\delta(s+\delta)). \tag{11}$$

From now on we consider the map $g=f^{k_\delta}$. Note that Λ is also a hyperbolic set of g. Evidently $W_\varepsilon^s(\Lambda)$ is forward invariant under g. It follows from the variational principle that $P_g(\phi_g^u) = k_\delta P_f(\phi_f^u)$; moreover $s_g = k_\delta s_f$. Thus it is sufficient to prove inequality (3) for g. We continue to use the notation $s, \phi^u, P(\phi^u)$, etc. for g instead of f. Let $x \in \Lambda$ and $k \in \mathbb{N}$. It follows from Proposition 4 that for ε sufficiently small,

$$P(\phi^u) = \lim_{k\to\infty} \frac{1}{k} \log(\mathrm{vol}(B(\Lambda, 2\varepsilon, k))), \tag{12}$$

where $B(\Lambda, 2\varepsilon, , k) = \bigcup_{x\in\Lambda} B(x, 2\varepsilon, k)$ (see (6)). From this we obtain that, if k is sufficiently large then

$$\mathrm{vol}(B(\Lambda, 2\varepsilon, k)) < \exp(k(P(\phi^u) + \delta)). \tag{13}$$

For all $k \in \mathbb{N}$ we define real numbers

$$r_k = \frac{\varepsilon}{\exp(s+\delta)^k}$$

and neighborhoods $B_k = B(W_\varepsilon^s(\Lambda), r_k)$ of $W_\varepsilon^s(\Lambda)$. Let $y \in B_k$. Then there exists $x \in W_\varepsilon^s(\Lambda)$ with $d(x, y) < r_k$. An elementary induction argument in combination with the mean-value theorem implies $d(g^i(x), g^i(y)) < \varepsilon$ for all $i \in \{0, \ldots, k-1\}$. Using (8) and making ε smaller if necessary, we can assure that x is contained in the local stable manifold of size ε of a point in Λ. It follows that $y \in B(\Lambda, 2\varepsilon, k)$. Hence $B_k \subset B(\Lambda, 2\varepsilon, k)$. Therefore, (13) implies that

$$\mathrm{vol}(B_k) < \exp(k(P(\phi^u) + \delta)) \tag{14}$$

for sufficiently large k. Let us recall that for $t \in [0, n]$ the t-dimensional upper Minkowski content of a relatively compact set $A \subset M$ is defined by

$$M^{*t}(A) = \limsup_{\rho\to 0} \frac{\mathrm{vol}(A_\rho)}{(2\rho)^{n-t}},$$

where $A_\rho = \{p \in M : \exists q \in A : d(p, q) \le \rho\}$, and vol denotes the volume induced by the Riemannian metric on M. Let $t \in [0, n]$ and $\rho_k = \frac{r_k}{2}$ for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned} M^{*t}(W_\varepsilon^s(\Lambda)) &= \limsup_{\rho\to 0} \frac{\mathrm{vol}(W_\varepsilon^s(\Lambda)_\rho)}{(2\rho)^{n-t}} \le \limsup_{k\to\infty} \frac{\mathrm{vol}(W_\varepsilon^s(\Lambda)_{\rho_k})}{(2\rho_{k+1})^{n-t}} \\ &\le \limsup_{k\to\infty} \frac{\mathrm{vol}(B_k)}{(r_{k+1})^{n-t}} \\ &\le \frac{\exp(s+\delta)^{n-t}}{\varepsilon^{n-t}} \lim_{k\to\infty} \left(k \exp(s+\delta)^{n-t} \exp(P(\phi^u) + \delta)\right). \end{aligned} \tag{15}$$

Let $t > n + \frac{P(\phi^u)+\delta}{s+\delta}$. Then $\exp(s+\delta)^{n-t}\exp(P(\phi^u)+\delta) < 1$. This implies $M^{*t}(W_\varepsilon^s(\Lambda)) = 0$. Hence $t \ge \overline{\dim}_B W_\varepsilon^s(\Lambda)$ (see [11]). Since δ can be chosen arbitrarily small, the result follows. □

Remark. We note that the idea of estimating the dimension of an invariant set by calculating the volume of neighborhoods of the set was introduced by the second author of this paper in [17] for estimating the dimension of an invariant set of a C^1 diffeomorphism. For a related result in the case of polynomial automorphisms of $\mathbb{C}^n$ see [15].

Applying Theorem 1 to f^{-1} we obtain an analogous result for the local unstable set

$$W_\varepsilon^u(\Lambda) = \{x \in M : \mathrm{dist}(f^{-k}(x), \Lambda) < \varepsilon \text{ for all } k \in \mathbb{N}\}.$$

Corollary 5. *Let $f : M \to M$ be a C^2 diffeomorphism on an n-dimensional Riemannian manifold, $\Lambda \subset M$ a locally maximal hyperbolic set, which is not a periodic orbit, such that $f|\Lambda$ is topologically mixing. Let $\phi^s = \log|\det Df|E^s|$. Then for sufficiently small $\varepsilon > 0$ we have*

$$\overline{\dim}_B W^u_\varepsilon(\Lambda) \leq n + \frac{P(\phi^s)}{s}, \tag{16}$$

where s is defined as in (2) *for the map f^{-1}.*

Note that if M is a surface, then the dimension of $W^{u/s}_\varepsilon(\Lambda)$ can be expressed using Bowen's formula. Namely,

$$t^{u/s} \stackrel{\text{def}}{=} \dim_H W^{u/s}(x) \cap \Lambda = \overline{\dim}_B W^{u/s}(x) \cap \Lambda$$

are independent of $x \in \Lambda$ (see [12]), and it is not too hard to see that

$$\dim_H W^{s/u}_\varepsilon(\Lambda) = \overline{\dim}_B W^{s/u}_\varepsilon(\Lambda) = t^{u/s} + 1. \tag{17}$$

The right equality in (17) can be shown using the fact that when $n = 2$ the holonomies are lipschitz continuous (see for instance [8]). On the other hand, the following example shows that the dimension of $W^s_\varepsilon(\Lambda)$ can be arbitrarily close to n.

Example 1. Let $B \subset \mathbb{R}^2$ be a unit square, and let $f : B \to \mathbb{R}^2$ be a linear horseshoe map with the expansion rate $\lambda^u > 2$ and the contraction rate $\lambda^s < \frac{1}{2}$, see Figure 1. It is well-known that $\Lambda = \{x \in B : f^k(x) \in B \text{ for all } k \in \mathbb{Z}\}$ is a locally maximal hyperbolic set of f, and that $f|\Lambda$ is topologically mixing. Moreover, we have

$$t^u = \frac{\log 2}{\log \lambda^u} \quad \text{and} \quad t^s = -\frac{\log 2}{\log \lambda^s}.$$

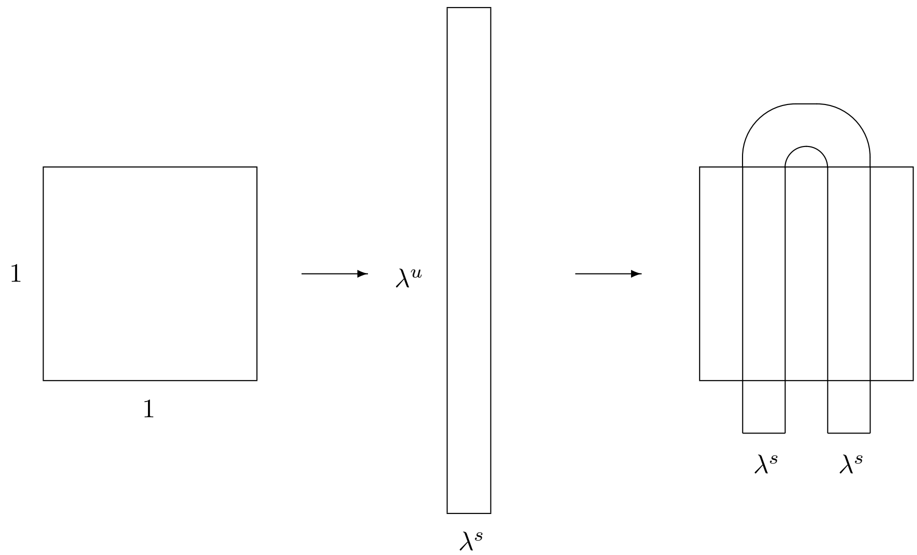

Figure 1. A Linear Horseshoe Map

As it was mentioned above, we also have $\dim_H W^{s/u}_\varepsilon(\Lambda) = \overline{\dim}_B W^{s/u}_\varepsilon(\Lambda) = t^{u/s} + 1$. Therefore, by choosing λ^u close to 2 (respectively λ^s close to $\frac{1}{2}$) we obtain the following.

Corollary 6. *For each $\varepsilon > 0$ there exists a linear horseshoe map of $\mathbb{R}^2$ such that* $\dim_H W_\varepsilon^{s/u}(\Lambda) = \overline{\dim}_B W_\varepsilon^{s/u}(\Lambda) > 2 - \varepsilon$.

We now apply Theorem 1 to obtain an upper bound for the Hausdorff dimension of the stable set of Λ.

Corollary 7. *Let $f : M \to M$ be a C^2-diffeomorphism, and Λ a locally maximal hyperbolic set of f, which is not a periodic orbit, such that $f|\Lambda$ is topologically mixing. Define $\phi^u = -\log|\det Df|E^u|$. Then*

$$\dim_H W^s(\Lambda) \le n + \frac{P(\phi^u)}{s}. \tag{18}$$

Proof. It follows from Theorem 1 (also using (10)) that $\dim_H W_\varepsilon^s(\Lambda) \le n + \frac{P(\phi^u)}{s}$. Obviously, $W_\varepsilon^s(x) \subset W_\varepsilon^s(\Lambda)$ for all $x \in \Lambda$, hence

$$\dim_H \bigcup_{x\in\Lambda} W_\varepsilon^s(x) \le n + \frac{P(\phi^u)}{s}. \tag{19}$$

It is a consequence of the shadowing lemma that

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x). \tag{20}$$

On the other hand, we have $W^s(x) = \bigcup_{k\in\mathbb{N}} f^{-k}(W_\varepsilon^s(f^k(x)))$. Together we obtain

$$W^s(\Lambda) = \bigcup_{k\in\mathbb{N}} f^{-k}\left(\bigcup_{x\in\Lambda} W_\varepsilon^s(x)\right). \tag{21}$$

The corollary now follows from equations (19), (21) and the fact that the Hausdorff dimension is stable with respect to countable unions. □

Remark. Since we have taken countable unions of sets whose box dimensions are uniformly bounded above by $n + \frac{P(\phi^u)}{s}$, we obtain the same upper bound for the packing dimension of $W^s(\Lambda)$.

We now provide the proofs of the corollaries stated in the introduction.

Proof of Corollary 2. If Λ is an attractor, then Λ can not be a repeller. Otherwise, we would have $W_\varepsilon^{u/s}(x) \subset \Lambda$ for all $x \in \Lambda$, in which case, since Λ has a local product structure, Λ would have non-empty interior. Therefore, Proposition 4 (or the analogous proposition for f^{-1}) implies that either $P(\phi^u) < 0$ or $P(\phi^s) < 0$. The result now follows from Theorem 1 and Corollary 5. □

Proof of Corollary 3. (i)⇒(iv) If (i) holds, then by Theorem 1, $P(\phi^u) = 0$. Therefore, (iv) follows from Proposition 4.
(iv) ⇒ (ii) is trivial.
(ii)⇒(iii) Assume $\dim_H W^s(\Lambda) = n$. Then it follows from Corollary 7 and Proposition 4 (i) that $P(\phi^u) = 0$. Since Λ is a locally maximal hyperbolic set on which f is topologically mixing, there exists a unique equilibrium measure μ_{ϕ^u} of the potential ϕ^u. This means that

$$0 = P(\phi^u) = h_{\mu_{\phi^u}}(f) + \int \phi^u d\mu_{\phi^u}, \tag{22}$$

where $h_{\mu_{\phi^u}}(f)$ denotes the measure-theoretic entropy of f with respect to μ_{ϕ^u}. Moreover, μ_{ϕ^u} is ergodic. Let $\lambda_1(x) < \cdots < \lambda_l(x)$ be the Lyapunov exponents of x with respect to f with multiplicities $m_1(x), \cdots, m_l(x)$. The fact that μ_{ϕ^u} is ergodic implies that the Lyapunov exponents and the multiplicities are constant μ_{ϕ^u}-almost everywhere. We denote the corresponding values by $\lambda_i(\mu_{\phi^u})$ and $m_i(\mu_{\phi^u})$. From the fact that Λ is a hyperbolic set it follows that

$$-\int \phi^u d\mu_{\phi^u} = \sum_{\lambda_i(\mu_{\phi^u})>0} \lambda_i(\mu_{\phi^u}) m_i(\mu_{\phi^u}). \tag{23}$$

Therefore, equations (22) and (23) imply that

$$h_{\mu_{\phi^u}}(f) = \sum_{\lambda_i(\mu_{\phi^u})>0} \lambda_i(\mu_{\phi^u}) m_i(\mu_{\phi^u}) = \int \sum_{\lambda_i(x)>0} \lambda_i(x) m_i(x)\, d\mu_{\phi^u}. \tag{24}$$

Hence μ_{ϕ^u} satisfies Pesin's entropy formula. Finally, from the work of Ledrappier, Strelcyn and Young in [9] and [10] we conclude that μ_{ϕ^u} is a SRB measure.
(iii)⇒(v) follows from [4] and [10].
Finally, (v)⇒(i) follows from (8) and (21). □

3. **Expanding maps.** In this section we consider a C^2 map f from a n-dimensional smooth Riemannian manifold M to itself. The map f is not assumed to be invertible. Let $\Lambda \subset M$ be a compact invariant set of f. We say that f is expanding on Λ if there exist $c > 0$ and $\lambda \in (1, \infty)$ such that for each $x \in \Lambda$:

$$\|Df^k(x)v\| \geq c\lambda^k \|v\| \quad \text{whenever} \quad v \in T_xM \text{ and } k \in \mathbb{N}.$$

Furthermore, we say that Λ is locally maximal if there exists an open neighborhood $U \subset M$ of Λ such that $\Lambda = \bigcap_{k\in\mathbb{N}} f^{-k}(U)$. If f is expanding on a locally maximal set Λ we say that Λ is a repeller of f. We shall always assume that f is expanding on Λ, Λ is locally maximal, and $f|\Lambda$ is topologically mixing. We define the function $\phi : \Lambda \to \mathbb{R}$ by $\phi(x) = -\log|\det Df(x)|$.

We start by proving the version of Proposition 4 for expanding maps.

Proposition 8. *If $\varepsilon > 0$ is small enough then*

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\text{vol}(B(\Lambda, \varepsilon, k)\right) \leq P(\phi) \leq 0 \tag{25}$$

Proof. We first show the right-hand side inequality of (25). Consider an ergodic invariant probability measure μ supported on Λ, and let $0 < \lambda_1(\mu) < \cdots < \lambda_l(\mu)$ be the Lyaponov exponents of μ. Denote by $m_i(\mu)$ the multiplicity of the Lyapunov exponent $\lambda_i(\mu)$. It follows from the Margulis-Ruelle inequality that

$$h_\mu(f) \leq \sum_i \lambda_i(\mu)\ m_i(\mu) = -\int \phi d\mu. \tag{26}$$

On the other hand, the variational principle states that

$$P(\phi) = \sup_\mu \left(h_\mu(f) + \int \phi d\mu \right), \tag{27}$$

where the supremum is taken over all (ergodic) invariant probability measures on Λ. Combining (26) and (27) yields $P(\phi) \leq 0$.

We now show the left-hand side inequality in (25). Fix some $\delta \leq \varepsilon$. Let $E_k(\delta)$ be a maximal (k, δ)-separated subset of Λ. Let $x \in \Lambda$; then $x \in B(y, \delta, k)$ for some $y \in E_k(\delta)$, because otherwise $E_k(\delta) \cup \{x\}$ would be (k, δ)-separated. Here $B(y, \delta, k)$ is defined as in (6). We conclude that $B(x, \varepsilon, k) \subset B(y, \delta + \varepsilon, k)$, and

$$B(\Lambda, \varepsilon, k) \subset \bigcup_{y\in E_k(\delta)} B(y, \delta + \varepsilon, k), \tag{28}$$

see (7) for the definition. For $x \in \Lambda$ and $k \in \mathbb{N}$ we define

$$S_k\phi(x) = \sum_{i=0}^{k-1} \phi(f^i(x)).$$

Analogously as in the case of diffeomorphisms (see [4], [5]), there exists $C_{\delta+\varepsilon} > 1$ such that if ε is small enough then

$$\mathrm{vol}(B(x, \delta + \varepsilon, k)) \leq C_{\delta+\varepsilon} \cdot \exp S_k\phi(x) \tag{29}$$

for all $x \in \Lambda$ and $k \in \mathbb{N}$. Therefore (28) and (29) imply that

$$\mathrm{vol}(B(\Lambda, \varepsilon, k)) \leq C_{\delta+\varepsilon} \cdot \sum_{y \in E_k(\delta)} \exp S_k\phi(y). \tag{30}$$

The map $f|\Lambda$ is expansive; therefore for δ small enough we have

$$P(\phi) = \limsup_{k\to\infty} \frac{1}{k} \log \left(\sum_{y \in E_k(\delta)} \exp(S_k\phi(y)) \right). \tag{31}$$

The result follows now by taking the logarithm, dividing by k and taking the upper limit in (30) and applying (31). □

We would like to point out that the proof of Proposition 8 is based on ideas of the corresponding proof of Bowen [4] for diffeomorphisms.

Quan and Zhu classified in [14] the invariant measures μ of a C^2 endomorphism f for which Pesin's entropy formula holds; they showed that μ has this condition if and only if μ satisfies the SRB property. This property is a generalized condition of an SRB measure for diffeomorphisms defined for the corresponding measure on the inverse limit map, see [14] for details. In particular, if f is an expanding map and μ is absolutely continuous with respect to Lebesgue, then Pesin's entropy formula holds, see [7].

Proposition 9. *$P(\phi) = 0$ if and only if f admits an invariant measure μ satisfying the SRB-property. In particular, this is the case when f has an invariant measure which is absolutely continuous with respect to Lebesgue.*

Proof. If $P(\phi) = 0$ then the same arguments as in the proof of Corollary 3 imply that there exists an invariant measure μ satisfying Pesin's formula; hence, μ has the SRB property. On the other hand, if μ has the SRB property then μ satisfies Pesin's entropy formula. It now follows from the variational principle and a similar argument as in the proof of Corollary 3 that $P(\phi) = 0$. □

Remark. It is easy to see that $P(\phi) = 0$ actually occurs. For example, if $f : S^1 \to S^1, f(z) = z^2$, then the entire manifold S^1 is a repeller. In this case the measure of maximal entropy (given by the distribution of the periodic points of f) is absolutely continuous with respect to Lebesgue.

We now present our main results for expanding maps.

Theorem 10. *Let f be a C^2 self map on an n-dimensional smooth Riemannian manifold M, and let Λ be a locally maximal repeller of f such that $f|\Lambda$ is topologically mixing. Then*

$$\overline{\dim}_B\Lambda \leq n + \frac{P(\phi)}{s}, \tag{32}$$

where s is defined as in (2).

Proof. The proof of the theorem is analogous to the proof of Theorem 1 just by replacing $W^s_\varepsilon(\Lambda)$ by Λ and applying Proposition 8 instead of Proposition 4. □

Corollary 11. *Let f be a C^2 self map of an n-dimensional smooth Riemannian manifold M, and let Λ be a repeller of f such that $f|\Lambda$ is topologically mixing. Then the following are equivalent.*

(i) $\overline{\dim}_B \Lambda = n$;
(ii) *f admits an invariant measure μ satisfying the SRB property.*

Proof. If $\overline{\dim}_B \Lambda = n$ then, by Theorem 10 and Proposition 8, $P(\phi) = 0$. Therefore, Proposition 9 implies that f admits an invariant measure μ satisfying the SRB property. On the other hand, if μ is an invariant measure satisfying the SRB property, then Pesin's entropy formula holds for μ, see [14]. It now follows from Theorem B of [7] that $\mathrm{vol}(\Lambda) > 0$, in particular, $\overline{\dim}_B \Lambda = n$. □

REFERENCES

[1] L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), no. 5, 871–927.
[2] L. Barreira, *Dimension estimates in nonconformal hyperbolic dynamics*, Nonlinearity **16** (2003), 1657-1672.
[3] R. Bowen, *A horseshoe with positive measure*, Invent. Math. **29** (1975), no. 3, 203–204.
[4] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470, Springer, 1975.
[5] R. Bowen, D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), 181–202.
[6] A. Franz, *Hausdorff dimension estimates for invariant sets with an equivariant tangent bundle splitting*, Nonlinearity 11 (1998), no. 4, 1063–1074.
[7] H.Y. Hu, *Pesin's entropy formula of expanding maps*, Adv. in Math. (China) 19 (1990), no. 3, 338–349.
[8] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of mathematics and its applications 54, Cambridge University Press, 1995.
[9] F. Ledrappier, J.M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory Dynam. Systems **2** (1982), 203–219.
[10] F. Ledrappier, L.-S. Young, *The metric entropy of diffeomorphisms, I, Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) 122 (1985), no. 3, 509–539.
[11] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge University Press, Cambridge, U.K., 1995.
[12] H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynam. Systems **3** (1983), 251–260.
[13] Ya. Pesin, *Dimension theory in dynamical systems: contemporary views and applications*, Chicago Lectures in Mathematics, Chicago University Press, 1997.
[14] M. Qian, S. Zhu, *SRB measures and Pesin's entropy formula for endomorphisms*, Trans. Amer. Math. Soc. 354 (2002), no. 4, 1453–1471.
[15] R. Shafikov, C. Wolf, *Filtrations, hyperbolicity and dimension for polynomial automorphisms of $\mathbb{C}^n$*, Michigan Math. J. 51 (2003), 631-649.
[16] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer, 1981.
[17] C. Wolf, *On the box-dimension of an invariant set*, Nonlinearity **14** (2001), 73 - 79.

Received for September 2003; revised August 2004.

E-mail address: shafikov@uwo.ca
URL: http://www.math.uwo.ca/~shafikov/
E-mail address: cwolf@math.wichita.edu
URL: http://www.math.wichita.edu/~cwolf/