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Some Aspects of Holomorphic Mappings: A Survey

Sergey Pinchuk[a], **Rasul Shafikov**[b], **and Alexandre Sukhov**[c]

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Abstract—This expository paper is concerned with the properties of proper holomorphic mappings between domains in complex affine spaces. We discuss some of the main geometric methods of this theory, such as the reflection principle, the scaling method, and the Kobayashi–Royden metric. We sketch the proofs of certain principal results and discuss some recent achievements. Several open problems are also stated.

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1. INTRODUCTION

This expository paper is dedicated to geometric properties of holomorphic mappings between domains in complex affine spaces (in general of different dimensions). The first results in this direction (mainly in complex dimension 2) are due to H. Poincaré, É. Cartan, and B. Segre. The rigidity of complex structures with boundary—one of the main phenomena of complex analysis in higher dimensions—was already discovered and studied in these classical works. The next major step in this theory was made in the 1970s with intensive investigation of the geometry of strictly pseudoconvex domains. Further progress concerns more general classes of domains (weakly pseudoconvex or not pseudoconvex at all). In this survey paper we try to present some of the

[a] Department of Mathematics, Indiana University, Rawles Hall, 831 E. Third St., Bloomington, IN 47405, USA.
[b] Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, Canada.
[c] UFR de Mathématiques, Université de Lille, Sciences et Technologies, 59655 Villeneuve d'Ascq Cedex, France.
E-mail addresses: pinchuk@indiana.edu (S. Pinchuk), shafikov@uwo.ca (R. Shafikov), sukhov@math.univ-lille1.fr (A. Sukhov).

main ideas in the development of the theory. Our presentation is certainly incomplete, as, regrettably, many important topics and results were not included in the scope of the paper. The interested reader may become acquainted with them using other monographs and expository papers [9, 10, 15, 29, 48, 68, 80, 81, 83, 87, 90, 91, 105, 113, 114, 125, 141, 147, 152, 153].

2. PRELIMINARIES

Denote by $z = (z_1, \dots, z_n)$ the standard complex coordinates in $\mathbb{C}^n$. We often use the (vector) notation $z = x + iy$ for the real and imaginary parts. Denote by $|z|$ the Euclidean norm of z and by $(z, w) = \sum_j z_j \overline{w_j}$ the Hermitian inner product. We also use the notation $\langle z, w\rangle = (z, \overline{w})$.

As usual, a *domain* Ω in $\mathbb{C}^n$ is a connected open subset of $\mathbb{C}^n$. Denote by $\partial\Omega$ the boundary of Ω. The unit ball of $\mathbb{C}^n$ is denoted by $\mathbb{B}^n = \{z \in \mathbb{C}^n\colon |z| < 1\}$, while for $n = 1$ we use the notation $\mathbb{D} := \mathbb{B}^1$ for the unit disc in $\mathbb{C}$. The ball $p + r\mathbb{B}^n$ of radius $r > 0$ centred at a point $p \in \mathbb{C}^n$ will be denoted by $\mathbb{B}^n(p, r)$. Another basic example of a domain in $\mathbb{C}^n$ is the unit polydisc $\mathbb{D}^n$ or, more generally, the polydisc $\mathbb{D}^n(p, r) := p + r\mathbb{D}^n$. Finally,

$$\mathbb{H} = \bigl\{z \in \mathbb{C}^n\colon\ 2\operatorname{Re} z_n + |z_1|^2 + \ldots + |z_{n-1}|^2 < 0\bigr\} \tag{2.1}$$

is an unbounded realization of the unit ball $\mathbb{B}^n$.

2.1. Classes of functions. Denote by $\mathcal{O}(\Omega)$ the class of holomorphic functions in a domain Ω. If Ω' is a domain in $\mathbb{C}^m$, we use the notation $\mathcal{O}(\Omega, \Omega')$ for the class of holomorphic mappings from Ω to Ω'.

For a positive integer k, $C^k(\Omega)$ denotes the space of C^k-smooth complex-valued functions in Ω. Also $C^k(\overline{\Omega})$ denotes the class of functions whose partial derivatives up to order k extend as continuous functions on $\overline{\Omega}$. If $s > 0$ is a real noninteger and k is its integer part, $C^s(\Omega)$ denotes the space of functions of class $C^k(\Omega)$ such that their partial derivatives of order k are (globally) Hölder continuous in Ω with the exponent $s - k$; these derivatives automatically satisfy the Hölder condition on $\overline{\Omega}$, so the notation $C^s(\overline{\Omega})$ for the same space of functions is also appropriate. Finally, we denote by $\mathrm{PSH}(\Omega)$ the class of plurisubharmonic functions on Ω.

Let Ω be a domain in $\mathbb{C}^n$ and $f\colon \Omega \to \mathbb{C}^N$ be a vector function (not necessarily holomorphic or smooth) on Ω. Let γ be a subset of the boundary $\partial\Omega$. The *cluster set* $C_\Omega(f; \gamma)$ of f on γ is defined as the set of all limit points of the sequences $\{f(z^k)\}$ in $\mathbb{C}^N$, where $\{z^k\}$ is any sequence in Ω converging to a point in γ. The cluster set $C_\Omega(f; \gamma)$ is empty if $\lim|f(z)| = +\infty$ as $z \to \gamma$. Note that a holomorphic map $f\colon \Omega \to \Omega'$ between two domains is proper if and only if the cluster set $C_\Omega(f; \partial\Omega)$ does not intersect Ω'. For bounded domains one can state this property in the equivalent form: $C_\Omega(f; \partial\Omega) \subset \partial\Omega'$.

2.2. Real submanifolds of complex spaces. A (closed) real submanifold E of a domain $\Omega \subset \mathbb{C}^n$ is of class C^s (respectively, real analytic) if for every point $p \in E$ there exists an open neighbourhood U of p and a map $\rho\colon U \to \mathbb{R}^d$ of the maximal rank $d < 2n$ and of class C^s (respectively, real analytic) such that $E \cap U = \rho^{-1}(0)$; then ρ is called a local defining (vector) function of E. The positive integer d is the real codimension of E. In the fundamental special case $d = 1$ we obtain the class of real hypersurfaces.

Let J denote the standard complex structure of $\mathbb{C}^n$. In other words, J acts on a vector V by multiplication by i. For every $p \in E$ the *holomorphic tangent space* $H_pE := T_pE \cap J(T_pE)$ is the maximal complex subspace of the tangent space T_pE of E at p. Clearly, $H_pE = \{V \in \mathbb{C}^n\colon \partial\rho(p)V = 0\}$. The complex dimension of H_pE is called the CR dimension of E at p; a manifold E is called a *CR* (*Cauchy–Riemann*) *manifold* if its CR dimension is independent of $p \in E$.

A real submanifold $E \subset \Omega$ is called *generic* (or *generating*) if the complex span of T_pM coincides with $\mathbb{C}^n$ for all $p \in E$. Note that every generic manifold of real codimension d is a CR manifold of

CR dimension $n-d$. A function $\rho=(\rho_1,\dots,\rho_d)$ defines a generic manifold if $\partial\rho_1\wedge\ldots\wedge\rho_d\neq 0$. Of special importance are the so-called *totally real manifolds*, i.e., submanifolds E for which $H_pE=\{0\}$ at every $p\in E$. A totally real manifold is generic if and only if its real dimension is equal to n; this is the maximal possible value for the dimension of a totally real manifold.

Let E be a generic manifold of real codimension d contained in the boundary $\partial\Omega$ of a domain Ω in $\mathbb{C}^n$. Our considerations are local. Consider tangent vector fields X_j, $j=1,\dots,n-d$, on E (of type $(1,0)$) which form a basis in the space of local sections of the holomorphic tangent bundle $H(E)$ near p. A C^1-smooth function f on E is called a CR function if it satisfies the following first-order PDE system on E:

$$X_jf=0,\qquad j=1,\dots,n-d. \tag{2.2}$$

These are the *tangential Cauchy–Riemann equations*. By Stokes' formula equations (2.2) can be rewritten in the equivalent form $[E](f\overline{\partial}\phi)=0$ for every test $(n,n-d)$-form ϕ on E; here $[E]$ denotes the current of integration over E. In this weak formulation the notion of a CR function can be extended to the class of continuous or locally integrable functions on E. If E is a hypersurface $(d=1)$ given by a defining function ρ with $\partial\rho/\partial z_n\neq 0$, then we can choose

$$X_j=\frac{\partial\rho}{\partial z_n}\frac{\partial}{\partial z_j}-\frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial z_n},\qquad j=1,\dots,n-1.$$

The following approximation theorem is due to Baouendi and Trèves [13].

Theorem 2.1. *Let M be a smooth generic manifold in $\mathbb{C}^n$ and $E\subset M$ be a smooth totally real manifold of dimension n. Then in a neighbourhood of any point $p\in E$, any CR function f of class C^s, $s\geq 0$, on M can be approximated in the C^s-norm on M by the sequence of holomorphic functions*

$$(\mathbf{1}_Ef)*\exp\{-k\langle z,z\rangle\},\qquad k=1,2,\dots,$$

where $\mathbf{1}_E$ is the characteristic function of E and the asterisk denotes the convolution operator.

2.3. Pseudoconvex and strictly pseudoconvex domains. Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that its boundary $\partial\Omega$ is a (compact) real hypersurface of class C^s in $\mathbb{C}^n$. Then there exists a C^s-smooth real function ρ in a neighbourhood U of the closure $\overline{\Omega}$ such that $\Omega=\{\rho<0\}$ and $d\rho|_{\partial\Omega}\neq 0$. We call such a function ρ a *global defining function*. If $s\geq 2$, one can consider the *Levi form* of ρ:

$$L(\rho,p,V)=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(p)V_j\overline{V}_k. \tag{2.3}$$

A bounded domain Ω with C^2 boundary is called *pseudoconvex* (respectively, *strictly pseudoconvex*) if $L(\rho,p,V)\geq 0$ (respectively, >0) for every $V\in H_p(\partial\Omega)$ (respectively, every nonzero $V\in H_p(\partial\Omega)$). This definition is equivalent to the general notion of pseudoconvexity in the Grauert–Oka sense: Ω is pseudoconvex if and only if it can be exhausted by a sequence of strictly pseudoconvex domains. Every strictly pseudoconvex domain Ω admits a global defining function which is strictly plurisubharmonic on a neighbourhood U of $\overline{\Omega}$. An analog of this property for pseudoconvex domains was established by Diederich and Fornaess [59].

Theorem 2.2. *Let Ω be a bounded pseudoconvex domain with C^s boundary, $s\geq 2$. Then there exist a C^s-smooth defining function ρ in a neighbourhood U of $\overline{\Omega}$ and a positive $\eta_0<1$ such that for any $0<\eta<\eta_0$ the function $\widehat{\rho}:=-(-\rho)^\eta$ is a strictly plurisubharmonic bounded exhaustion function for Ω (i.e., $\widehat{\rho}\colon\Omega\to(0,a)$ is a proper map for some $a>0$).*

The famous example of the so-called worm domain due to the same authors [58] shows that there exist smoothly bounded pseudoconvex domains without a plurisubharmonic defining function.

Let Γ be a real hypersurface of class C^2 in $\mathbb{C}^n$. One can view every holomorphic tangent space $H_p\Gamma$ as an element of the Grassmannian $G(n-1,n)$ of hyperplanes in $\mathbb{C}^n$. Then the holomorphic tangent bundle $H(\Gamma)$ can be viewed as a real submanifold of dimension $2n-1$ of the complex manifold $\mathbb{C}^n \times G(n-1,n)$ of complex dimension $2n-1$. We call it the *projectivization of the holomorphic tangent bundle* and denote it by $\mathbb{P}H(\Gamma)$. The following statement, due to Webster [157], is easy to check in local coordinates.

Lemma 2.3. *Γ has a nondegenerate Levi form if and only if $\mathbb{P}H(\Gamma)$ is a totally real manifold in $\mathbb{C}^n \times G(n-1,n)$.*

2.4. Kobayashi–Royden pseudometric. Let z be a point of a domain Ω and V be a tangent vector at z. The infinitesimal Kobayashi–Royden pseudometric $F_\Omega(z,V)$ (the "length" of the vector V) is defined as

$$F_\Omega(z,V) = \inf\biggl\{\lambda > 0\colon\ \exists\, h \in \mathcal{O}(\mathbb{D},\Omega) \ \text{with}\ h(0)=z,\ h'(0)=\frac{V}{\lambda}\biggr\}. \tag{2.4}$$

This is a nonnegative upper semicontinuous function on the tangent bundle of Ω; its integrated form coincides with the usual Kobayashi distance. The Kobayashi–Royden metric is decreasing under holomorphic mappings: if $f\colon \Omega \to \Omega'$ is a holomorphic mapping between two domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, then

$$F_{\Omega'}(f(z), df(z)V) \le F_\Omega(z,V). \tag{2.5}$$

In fact, this is the largest metric in the class of (properly normalized) infinitesimal metrics that are decreasing under holomorphic mappings. It is easy to obtain an upper bound on F_Ω. Indeed, let $\mathbb{B}^n(z,R)$ with $R = \operatorname{dist}(z,\partial\Omega)$ be the ball contained in Ω. It follows by the holomorphic decreasing property applied to the natural inclusion $\iota\colon \mathbb{B}^n(z,R) \to \Omega$ that the Kobayashi–Royden metric of this ball is greater than F_Ω. This gives the upper bound

$$F_\Omega(z,V) \le \frac{C|V|}{\operatorname{dist}(z,\partial\Omega)}. \tag{2.6}$$

Lower bounds require considerably more subtle analysis. Some general estimates can be obtained using plurisubharmonic functions. Sibony [139] proposed an approach based on the following Schwarz-type lemma for subharmonic functions. For a domain Ω and $z \in \Omega$, denote by $S_z(\Omega)$ the class of functions $u\colon \Omega \to [0,1]$ such that $u(z)=0$, u is of class C^2 in a neighbourhood of z, and $\log u$ is a plurisubharmonic function in Ω.

Lemma 2.4. *Let $u \in S_0(\mathbb{D})$. Then*

(a) *$u(\zeta) \le |\zeta|^2$ for $\zeta \in \mathbb{D}$; the equality holds at some point different from 0 if and only if $u(\zeta)$ is identically equal to $|\zeta|^2$;*

(b) *$\Delta u(0) \le 4$, with equality if and only if $u(\zeta) = |\zeta|^2$ for every $\zeta \in \mathbb{D}$.*

Consider an infinitesimal pseudometric P_Ω defined by

$$P_\Omega(z,V) = \sup\bigl\{L(u,z,V)^{1/2}\colon\ u \in S_z(\Omega)\bigr\}. \tag{2.7}$$

This pseudometric is locally bounded on the tangent bundle by Lemma 2.4(a) and is decreasing under holomorphic mappings; hence

$$P_\Omega \le F_\Omega.$$

To obtain the estimate from below for Sibony's metric, it suffices to construct a function $u \in S_z(\Omega)$ with controlled Levi form. For example, this leads to the following

Proposition 2.5. *Let Ω be a domain in $\mathbb{C}^n$ and ρ be a negative C^2-smooth plurisubharmonic function in Ω. Suppose that the partial derivatives of ρ are bounded on Ω and there exists a constant $C_1 > 0$ such that*

$$L(\rho, z, V) \geq C_1 |V|^2 \tag{2.8}$$

for all z and V. Then there exists a constant $C_2 > 0$, depending only on the C^2-norm of ρ, such that

$$P_\Omega(z, V) \geq C_2 \biggl(C_1^2 \frac{|\langle \partial\rho(z), V\rangle|^2}{|\rho(z)|^2} + C_1 \frac{|V|^2}{|\rho(z)|^2} \biggr). \tag{2.9}$$

Note that ρ is not assumed to be a defining function of Ω, although this special case is particularly important in applications. The original argument of Sibony assumes that Ω is globally bounded, but this condition can be dropped. In fact, estimate (2.9) holds on an open subset of Ω where (2.8) is satisfied. Therefore, it can be used in order to localize the Kobayashi–Royden metric. Note also that Ω is not assumed to be bounded or hyperbolic (see [25, 26, 49, 142, 143]). In particular, this leads to the following result (see [49]).

Proposition 2.6. *Let Ω be a domain in $\mathbb{C}^n$, ρ be a plurisubharmonic function in Ω with $E = \rho^{-1}(0)$, and let $f\colon \mathbb{D} \to \Omega^+ = \{\rho \geq 0\}$ be a bounded holomorphic mapping such that the cluster set $C_{\mathbb{D}}(f, \gamma)$ on an open arc $\gamma \subset \partial\mathbb{D}$ is contained in E. Assume that for a certain point $\zeta \in \gamma$ the cluster set $C_{\mathbb{D}}(f, \zeta)$ contains a point $p \in E$ such that, for some $\varepsilon > 0$, the function $\rho(z) - \varepsilon|z|^2$ is plurisubharmonic in a neighbourhood of p. Then f extends to a $1/2$-Hölder continuous mapping in a neighbourhood of ζ in $\mathbb{D} \cup \gamma$.*

The proof is based on estimate (2.9) in a tube neighbourhood of E of the form $\rho < \delta$ with small $\delta > 0$. A special case useful for applications arises when E is a totally real manifold: indeed, such a manifold can be represented as the zero locus of a nonnegative strictly plurisubharmonic function (see [96]).

2.5. Some properties of holomorphic functions near real manifolds. Analytic discs form an important special class of holomorphic mappings. Recall that an *analytic* (or *holomorphic*) *disc* in $\mathbb{C}^n$ is a holomorphic mapping $f\colon \mathbb{D} \to \mathbb{C}^n$. The most interesting case arises when analytic discs have some boundary regularity (at least, are continuous on $\overline{\mathbb{D}}$). The restriction $f\colon \partial\mathbb{D} \to \mathbb{C}^n$ is called the boundary of the analytic disc f. We say that a disc f is attached or glued to a subset K of $\mathbb{C}^n$ if $f(\partial\mathbb{D}) \subset K$.

Let E be a generic submanifold in a domain $\Omega \subset \mathbb{C}^n$ defined as $\{\rho = (\rho_1, \dots, \rho_d) = 0\}$. The *wedge* $W(\Omega, E)$ in Ω *with the edge* E is the domain

$$W(\Omega, E) = \bigl\{ z \in \Omega\colon\ \rho_j(z) < 0,\ j = 1, \dots, d \bigr\}.$$

One can also consider a more general class of domains if one fixes an open (convex) cone K in $\mathbb{R}^d$ and defines a wedge-type domain by the condition $\{z \in \Omega\colon \rho(z) \in K\}$. However, in many cases the study of holomorphic functions on such domains can be reduced to that on the simpler wedges $W(\Omega, E)$. For $\delta > 0$ we also consider a δ-"truncated" wedge

$$W_\delta(\Omega, E) = \Biggl\{ z \in \Omega\colon\ \rho_j(z) - \delta \sum_{k \neq j} \rho_k < 0,\ j = 1, \dots, d \Biggr\} \subset W(\Omega, E).$$

The complexification of a real analytic parametrization of a totally real submanifold yields the following result.

Proposition 2.7. *Let E be a real analytic totally real submanifold of dimension n in $\mathbb{C}^n$. For every point $p \in E$ there exists an open neighbourhood Ω in $\mathbb{C}^n$ and a holomorphic embedding $\Phi\colon \Omega \to \mathbb{C}^n$ such that $\Phi(p) = 0$ and $\Phi(E \cap \Omega) = \mathbb{R}^n \cap \Phi(\Omega)$.*

This proposition simplifies many aspects of complex analysis near real analytic totally real submanifolds of maximal dimension. If E is merely smooth, then a more subtle result holds: there exists a diffeomorphism Φ that takes E to $\mathbb{R}^n$ and is such that $\overline{\partial}\Phi$ vanishes to infinite order on E.

In the study of totally real submanifolds the following *gluing disc argument* is often quite helpful. It was introduced in [117] and then used by many authors. Without loss of generality, we may assume that in a neighbourhood Ω of the origin a smooth totally real manifold E is defined by the equation $x = r(x, y)$, where a smooth vector function $r = (r_1, \dots, r_n)$ satisfies the conditions $r_j(0) = 0$ and $\nabla r_j(0) = 0$. Fix a positive noninteger s and consider the Hilbert transform $H\colon u \to H(u)$ for a real function $u \in C^s(\partial\mathbb{D})$. It is uniquely defined by the conditions that the function $u + iH(u)$ is the trace of a function holomorphic on $\mathbb{D}$ and the integral average of $H(u)$ over the circle is equal to 0. This is a classical linear singular integral operator; it is bounded on the space $C^s(\partial\mathbb{D})$. Let $S^+ = \{e^{i\theta}\colon \theta \in [0, \pi]\}$ and $S^- = \{e^{i\theta}\colon \theta \in (\pi, 2\pi)\}$ be the semicircles. Fix a C^∞-smooth real function ψ_j on $\partial\mathbb{D}$ such that $\psi_j|S^+ = 0$ and $\psi_j|S^- < 0$, $j = 1, \dots, n$. Set $\psi = (\psi_1, \dots, \psi_n)$. Consider the *generalized Bishop equation*

$$u(\zeta) = r\bigl(u(\zeta), H(u)(\zeta) + c\bigr) + t\psi(\zeta), \qquad \zeta \in \partial\mathbb{D}, \tag{2.10}$$

where $c \in \mathbb{R}^n$ and $t = (t_1, \dots, t_n)$, $t_j \ge 0$, are real parameters. It follows by the implicit function theorem that this equation admits a unique solution $u(c, t) \in C^s(\partial\mathbb{D})$ depending smoothly on the parameters (c, t). Consider now the analytic discs $f(c, t)(\zeta) = P_{\mathbb{D}}(u(c, t)(\zeta) + iH(u(c, t))(\zeta))$, where $P_{\mathbb{D}}$ denotes the Poisson operator of harmonic extension to $\mathbb{D}$. The map $(c, t) \mapsto f(c, t)(0)$ (the centres of discs) is of class C^s. Every disc is attached to E along the upper semicircle. It is easy to see that this family of discs fills the wedge $W_\delta(\Omega, E)$ when $\delta > 0$ and a neighbourhood Ω of the origin are chosen small enough. Indeed, this is immediate when the function r vanishes identically (i.e., $E = i\mathbb{R}^n$), while the general case follows by a small perturbation argument.

This construction of gluing analytic discs is flexible enough and has several applications. As an example we prove a version of the *edge-of-the-wedge theorem* following [1, 148].

Consider the generic manifolds $E_j = \{z \in \Omega\colon \rho_k(z) = 0,\ k \ne j,\ k = 1, \dots, n\}$ of dimension $n + 1$. On the unit circle we consider the open arcs S_j, $j = 1, \dots, n$, bounded by the points $\{e^{2\pi i j/n}, j = 0, \dots, n - 1\}$. Let ψ_j be C^∞-smooth functions on $\partial\mathbb{D}$ such that $\psi_j|S_j < 0$ and $\psi_j|(\partial\mathbb{D} \setminus S_j) = 0$, $j = 1, \dots, n$. Equation (2.10) admits a solution in $C^s(\partial\mathbb{D})$ which depends smoothly on the parameters (c, t) in a neighbourhood of the origin in $\mathbb{R}^{2n}$ (note that t_j are not assumed to be positive here). Every analytic disc from the family $f(c, t)(\zeta)$ obtained as above has the boundary attached to the union $\bigcup_j E_j$. Furthermore, their centres $f(c, t)(0)$ fill a neighbourhood of the origin in $\mathbb{C}^n$. Indeed, the map $(c, t) \mapsto f(c, t)(0)$ has the maximal rank $2n$ in a neighbourhood of the origin (this is obvious when $r = 0$ and hence remains true under small perturbations). In combination with the approximation result (Theorem 2.1) we obtain

Proposition 2.8. *Let f be a continuous CR function on $\bigcup_j E_j$. Then f extends holomorphically to a neighbourhood of E in $\mathbb{C}^n$.*

Indeed, by the maximum principle (applied along every analytic disc) the approximating family of holomorphic functions converges in a neighbourhood of the origin.

As a corollary we get the edge-of-the-wedge theorem (for a more general result see [119]). Introduce the domains $\Omega^+ = \{z \in \Omega\colon \rho_j > 0,\ j = 1, \dots, n\}$ and $\Omega^- = \{z \in \Omega^-\colon \rho_j(z) < 0,\ j = 1, \dots, n\}$.

Corollary 2.9. *Let f^+ and f^- be functions holomorphic on the wedges Ω^+ and Ω^-, respectively, and continuous up to the edge E. If f^+ and f^- coincide on E, then they extend to a holomorphic function in a neighbourhood of E.*

In combination with Proposition 2.7 and the Schwarz reflection principle, this immediately gives the following simple multidimensional version of this principle.

Proposition 2.10. *Let E and E' be real analytic totally real manifolds of dimension n and N in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively. Suppose that $f\colon W(\Omega,E)\to\mathbb{C}^N$ is a holomorphic mapping continuous on $W_\delta(\Omega,E)\cup E$ for some $\delta>0$ and such that $f(E)\subset E'$. Then f extends holomorphically to a neighbourhood of E.*

A smooth version of this result also holds but requires some additional technical tools.

Proposition 2.11. *Let $W(\Omega,E)$ be a wedge in $\mathbb{C}^n$ with a C^∞-smooth totally real edge E of dimension n. Suppose that $f\colon W(\Omega,E)\to\mathbb{C}^N$ is a holomorphic mapping such that the cluster set $C_{W(\Omega,E)}(f;E)$ is contained in a C^∞-smooth totally real manifold E' of dimension N. Then for every $\delta>0$ the mapping f extends to $W_\delta(\Omega;E)\cup E$ as a C^∞-smooth mapping.*

We sketch the proof based on the ideas of Pinchuk and Hasanov [127] (for details see [56]). The first step is to establish the result for $n=1$, i.e., when f is an analytic disc. This is a combination of Proposition 2.6 and Chirka's boundary regularity theorem for analytic discs [46]. The second step is to apply the above construction of filling $W(\Omega,E)$ with analytic discs glued to E along the upper semicircle. Since the Hölder constants are uniform with respect to the parameters, this implies the Hölder continuity of f up to E. In the last step we use the smooth version of Proposition 2.7. Let Φ and Ψ be (local) diffeomorphisms which take E and E' to $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, and such that $\overline{\partial}\Phi$ and $\overline{\partial}\Psi$ vanish to infinite order on E and E', respectively. We can apply the usual reflection principle to the mapping $\Psi\circ f\circ\Phi^{-1}$. This gives two functions in the opposite wedges with the edge $\mathbb{R}^n$. The functions are continuous up to $\mathbb{R}^n$, coincide there, and have the property that the $\overline{\partial}$-part of their differential vanishes to a suitable order on $\mathbb{R}^n$. But then these functions are C^∞-smooth up to the edge $\mathbb{R}^n$. This is a very special case of the general elliptic regularity of the $\overline{\partial}$-operator. In our case it can be directly proved by slicing with complex linear discs and using regularity of the Cauchy integral transform $f\mapsto(2\pi i)^{-1}f*(1/\zeta)$ on $\mathbb{D}$.

3. GEOMETRY OF REAL ANALYTIC HYPERSURFACES

Real hypersurfaces in $\mathbb{C}^n$, $n>1$, have nontrivial geometry induced by the complex structure of the ambient space. This is the main reason for rigidity of holomorphic mappings between domains in $\mathbb{C}^n$. In this section we describe classical methods used in the investigation of rigidity properties of holomorphic mappings near boundaries of domains.

3.1. Complexification, Segre varieties, and differential equations. We first introduce an important family of local biholomorphic invariants of a Levi nondegenerate real analytic hypersurface Γ in $\mathbb{C}^n$, $n>1$. This is a family of complex hypersurfaces called *Segre varieties* of Γ. One can view them as (the graphs of) solutions of a holomorphic second-order PDE system with a completely integrable prolongation to the space of 1-jets. When $n=2$, such a system becomes a second-order holomorphic ODE and the Segre family consists of complex curves. The biholomorphic maps of Γ are precisely the Lie symmetries of its Segre family. Thus, the geometry of real analytic hypersurfaces is closely related to the geometry of holomorphic ODEs and PDEs. This fundamental correspondence, discovered by Segre [136], inspired É. Cartan [37] to study the geometry of real hypersurfaces in $\mathbb{C}^2$ in analogy with the geometry of a second-order ODE developed by the school of S. Lie (see [146]). The approach of É. Cartan is very different and is based on his equivalence method for Pfaffian systems.

All considerations of this section are local, so the results should be understood in terms of the germs of the analytic objects involved. To simplify the notation, we will not use the language of germs, so the reader should keep in mind the locality assumption.

Let Γ be a real analytic hypersurface in a neighbourhood of the point $0 \in \Gamma$ in $\mathbb{C}^n$. Then $\Gamma = \{z\colon \rho(z,\overline{z}) = 0\}$, where ρ is a local defining real analytic function. For w close enough to the origin we consider the complex hypersurface

$$Q_w = \{z\colon \rho(z,\overline{w}) = 0\}. \tag{3.1}$$

This hypersurface is called the *Segre variety* (associated with Γ) of the point w. The collection of all Segre varieties is called the *Segre family* of Γ. More generally, if Γ is any real analytic set defined as the zero set of the vector function ρ, its Segre varieties are complex analytic subsets of $\mathbb{C}^n$ also defined by (3.1).

The following basic properties of the Segre family can be easily checked:

(i) $z \in Q_z$ if and only if $z \in \Gamma$;

(ii) $z \in Q_w$ if and only if $w \in Q_z$;

(iii) if F is a holomorphic mapping in a neighbourhood of the origin such that $F(\Gamma) \subset \Gamma'$, where Γ' is another real analytic hypersurface, then $F(Q_w) \subset Q'_{F(w)}$, where $Q'_\bullet$ denotes the Segre family of Γ'.

Property (iii) means that the Segre family is invariant with respect to biholomorphic mappings. In the one-dimensional case Segre varieties are points and so property (iii) becomes the classical Schwarz reflection principle. In higher dimensions this property leads to far-reaching consequences. For applications it is convenient to state property (iii) in a more general form.

Lemma 3.1. *Let $\Gamma \ni 0$ be a real analytic hypersurface in $\mathbb{C}^n$ and $\Gamma' \ni 0$ be a real analytic subset of $\mathbb{C}^N$. Let F be a holomorphic mapping such that $F(0) = 0$ and $F(\Gamma) \subset \Gamma'$. Then $F(Q_w) \subset Q'_{F(w)}$, where $Q'_\bullet$ denotes the Segre family of Γ'.*

For the proof let $\Gamma' = \{z' \in \mathbb{C}^N\colon \phi_j(z',\overline{z}') = 0,\ j = 1,\dots,d\}$. Then $\phi_j(F(z),\overline{F(z)}) = 0$ whenever $\rho(z,\overline{z}) = 0$. Therefore, $\phi_j(F(z),\overline{F(z)}) = \lambda_j(z,\overline{z})\rho(z,\overline{z})$, where λ_j is a real analytic function in a neighbourhood of the origin. It follows that $\phi_j(F(z),\overline{F(w)}) = \lambda_j(z,\overline{w})\rho(z,\overline{w})$ for (z,w) close to the origin in $\mathbb{C}^n \times \mathbb{C}^N$. This proves the lemma.

We now draw a connection between the complex geometry of real analytic hypersurfaces and the geometry of analytic differential equations and projective connections. The main idea is that the Segre family is a general set of solutions of some second-order PDE system (or a single second-order ODE when $n = 2$).

Let us discuss in some detail the case of dimension 2. We begin with the basic example of $\Gamma = \{z_2 + \overline{z}_2 + z_1\overline{z}_1 = 0\}$, an unbounded realization of the unit sphere in $\mathbb{C}^2$. The Segre family has the form $Q_w = \{z \in \mathbb{C}^2\colon z_2 + \overline{w}_2 + z_1\overline{w}_1 = 0\}$. This is just the family of all complex lines in $\mathbb{C}^2$ (except the "vertical" lines $z_1 = \mathrm{const}$). We view every Q_w as the graph of a complex affine function $z_2 = h(z_1)$ that depends on two complex parameters w_1 and w_2. We treat z_1 as an independent variable and z_2 as a dependent one. Then the Segre family is the set of graphs of all solutions of the ordinary differential equation $\ddot{z}_2 = 0$. In the general case we can assume that $M = \{z\colon \operatorname{Re} z_2 = \phi(z_1,\overline{z}_1,\operatorname{Im} z_2)\}$, where $\nabla\phi(0) = 0$. By the implicit function theorem, $Q_w = \{z\colon z_2 = h(z_1,\overline{w}_1,\overline{w}_2)\}$ for some holomorphic function h. Again we view z_2 as the dependent variable and z_1 as the independent one. Applying the chain rule, we obtain $d^j z_2/dz_1^j = (\partial^j h/\partial z_1^j)(z_1,\overline{w}_1,\overline{w}_2)$, $j = 1, 2$. By the implicit function theorem, we represent the parameters (w_1,w_2) as functions of $(z_1,z_2,\dot{z}_2)$ and obtain the holomorphic ODE

$$\ddot{z}_2 = F(z_1,z_2,\dot{z}_2). \tag{3.2}$$

The Segre family of the real hypersurface Γ is precisely the set of the graphs of the solutions of (3.2).

The invariance property (iii) of the Segre family means that a biholomorphism f of Γ sends the graph of a solution of (3.2) to the graph of another solution. But this means precisely that f

is a (point) Lie symmetry of equation (3.2). Therefore, one can apply the classical theory of Lie symmetries in order to study biholomorphisms of real analytic hypersurfaces.

As an example, consider a holomorphic differential equation

$$(S): \qquad \ddot{u} = F(x, u, \dot{u}),$$

where x denotes an independent complex variable and F is a holomorphic function. A *symmetry group* $\mathrm{Sym}(S)$ of equation (S) is a (maximal) local (Lie) group G acting on a domain in $\mathbb{C}^2$ in such a way that the following holds: for every solution $u(x)$ of equation (S) and every $g \in G$ the image (if defined) of the graph of u under g is the graph of some solution of equation (S), which we denote by $g_* u$. A holomorphic vector field

$$X = \theta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{3.3}$$

is called an *infinitesimal Lie symmetry* of equation (S) if it belongs to the Lie algebra of $\mathrm{Sym}(S)$, i.e., generates a one-parameter group of point Lie symmetries of equation (S). Denote by $j_x^m(u)$ the m-jet of u at x. In the most important case $m = 2$ we set $u_1 = u_x$ and $u_{11} = u_{xx}$. Then $j_x^2(u) = (x, u, u_1, u_{11})$, and so (x, u, u_1, u_{11}) are natural coordinates on this jet space.

Every point Lie symmetry g canonically extends to $J^m(1,1)$ as a biholomorphic mapping $g^{(m)}$ defined as follows: $g^{(m)}$ associates to $J_x^m(u)$ the jet $j_{u(x)}^m(g_* u)$. In particular, a one-parameter group of symmetries generated by a vector field X lifts to $J^m(1,1)$. A vector field $X^{(m)}$ on $J^m(1,1)$ which generates this lift is called the prolongation of order m of X. The classical Lie theory provides powerful tools for the study of Lie symmetries, which are particularly convenient in the infinitesimal case. For $m = 2$ we have

$$X^{(2)} = X + \eta_1 \frac{\partial}{\partial u_1} + \eta_{11} \frac{\partial}{\partial u_{11}}. \tag{3.4}$$

The general Lie theory gives the following expressions for the coefficients:

$$\eta_1 = \eta_x + (\eta_u - \theta_x) u_1 - \theta_u (u_1)^2, \tag{3.5}$$

$$\eta_{11} = \eta_{xx} + (2\eta_{xu} - \theta_{xx}) u_1 + (\eta_{uu} - 2\theta_{xu})(u_1)^2 - \theta_{uu}(u_1)^3 + (\eta_u - 2\theta_x) u_{11} - 3\theta_u u_1 u_{11}. \tag{3.6}$$

Equation (S) defines a complex hypersurface (S_2) in $J^2(1,1)$ by the equation $u_{11} = F(x, u, u_1)$. The fundamental principle of the Lie theory states that X is an infinitesimal symmetry of equation (S) if and only if the vector field $X^{(2)}$ is tangent to (S_2), that is,

$$X^{(2)}(u_{11} - F(x, u, u_1)) = 0 \qquad \text{for} \quad (x, u, u_1, u_{11}) \in (S_2). \tag{3.7}$$

Consider the expansion $F(x, u, u_1) = \sum_{\nu \ge 0} f_\nu(x, u)(u_1)^\nu$. Plugging it into (3.7) and comparing the coefficients of the powers of u_1, we obtain a system of PDEs of the form

$$LD^2(\theta, \eta) = G(x, u, D^1(\theta, \eta)).$$

Here D^j denotes the set of the partial derivatives of the map (θ, η) of order j, G is an analytic function, and L is a matrix with constant coefficients. Applying the partial derivatives with respect to x and u to this system, after a direct computation we obtain

$$D^3(\theta, \eta) = H\big(x, u, D^1(\theta, \eta), D^2(\theta, \eta)\big) \tag{3.8}$$

for some analytic function H. This implies that every infinitesimal Lie symmetry of equation (S) is determined by its second-order jet at a given point. In particular, $\dim \mathrm{Sym}(S) \le 8$.

Consider again equation (3.2) describing the Segre family of Γ. The group of local biholomorphisms of Γ is embedded into the symmetry group of (3.2) as a totally real subgroup of maximal dimension (see [144] for more details). As a consequence we find that the dimension of the real Lie group of biholomorphisms of Γ is bounded above by 8.

In higher dimensions ($n > 2$) the Segre family of a Levi nondegenerate real analytic hypersurface Γ is described by a PDE system

$$u_{x_i x_j} = F_{ij}(x, u, u_x), \tag{3.9}$$

where $x \in \mathbb{C}^{n-1}$ and u_x denotes the set of the first-order partial derivatives of the function $u = u(x)$. This system is completely integrable: its lift to the space of first-order jets is a first-order PDE system which satisfies the Frobenius integrability conditions. For more details we refer the reader to papers [114, 144], which are devoted to the study of Lie symmetries of PDE systems and Segre families.

3.2. Equivalence problem for real hypersurfaces: Moser's approach. Let Γ be a real analytic strictly pseudoconvex hypersurface in $\mathbb{C}^n$ containing the origin. All consideration will be local. We use the notation $z = ('z, z_n)$, $'z \in \mathbb{C}^{n-1}$. By the implicit function theorem (after a permutation of coordinates), Γ can be written as the graph

$$y_n = F('z, '\overline{z}, x_n) \tag{3.10}$$

of a real analytic function F. Given a point $p \in \Gamma$, consider a real analytic curve γ in Γ with a parametrization $z = z(\tau)$, $\tau \in (-\tau_0, \tau_0)$. Assume that γ passes through p in a noncomplex tangential direction, i.e., $z(0) = p$ and the vector $\dot{z}(0)$ is not in $H_p\Gamma$.

In a neighbourhood of p there exists a biholomorphic map $z^* = h(z)$ taking the curve γ to the real interval $'z = 0$, $z_n = \tau$ (we drop the asterisk for simplicity), that is, $h(z(\tau)) = ('0, \tau)$. Furthermore, in the new coordinates Γ is given by the equation

$$y_n = |'z|^2 + \sum_{k,l \ge 2} F_{kl}('z, '\overline{z}, x_n), \tag{3.11}$$

where F_{kl} are real homogeneous polynomials of degree k in $'z$ and l in $'\overline{z}$ with coefficients analytic in x_n.

Of course, such a change of coordinates is not unique. First of all this is due to the freedom in the choice of the curve γ and its parametrization. Moreover, consider a transformation

$$('z, z_n) \mapsto (U(z_n)'z, z_n). \tag{3.12}$$

Here $z_n \mapsto U(z_n)$ is an $(n-1) \times (n-1)$ nondegenerate holomorphic matrix function of z_n which is unitary for $z_n = \tau$. This transformation fixes γ and preserves (3.11).

Using tensor notation, we write

$$F_{kl} = \sum_{1 \le \alpha_\nu, \beta_\mu \le n-1} a_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l} z_{\alpha_1} \dots z_{\alpha_k} \overline{z}_{\beta_1} \dots \overline{z}_{\beta_l}.$$

Here we assume that the coefficients $a_{\alpha\beta}$ do not change under permutations of indices α_ν and β_μ. For $k, l \ge 1$, we put

$$\operatorname{tr} F_{kl} = \sum b_{\alpha_1 \dots \alpha_{k-1} \beta_1 \dots \beta_{l-1}} z_{\alpha_1} \dots z_{\alpha_{k-1}} \overline{z}_{\beta_1} \dots \overline{z}_{\beta_{l-1}}$$

with

$$b_{\alpha_1 \dots \alpha_{k-1} \beta_1 \dots \beta_{l-1}} = \sum_{\alpha_k = \beta_l} a_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l}.$$

Moser [45] proved that after a biholomorphic change of coordinates one can additionally achieve in (3.11) the conditions

$$\operatorname{tr} F_{22} = (\operatorname{tr})^2 F_{32} = (\operatorname{tr})^3 F_{33} = 0. \tag{3.13}$$

Representation (3.11), (3.13) is called the (*Moser*) *normal form* of Γ. A real analytic curve γ which in the normal form has the equation $'z = 0$, $y_n = 0$ is called a *chain.*

Conditions (3.13) can be interpreted geometrically. First we note that for a given point $p \in \Gamma$ there exists a unique chain passing through p in a prescribed noncomplex tangential direction. Furthermore, if Γ is given by equation (3.11), then the line $'z = 0$, $y_n = 0$ is a chain if and only if $(\operatorname{tr})^2 F_{32} = 0$. Let now A be a unitary $(n-1) \times (n-1)$ matrix. There exists a unique mapping (3.12) such that $U(0) = A$ and in the new coordinates $\operatorname{tr} F_{22} = 0$. This matrix A can be viewed as a new choice of an orthonormal basis in $H_0\Gamma$. Finally, consider an admissible reparametrization $('z, z_n) \mapsto (\sqrt{\dot q(z_n)}\,'z, q(z_n))$ of γ. Here $q(0) = 0$ and $q(\overline{z}_n) = \overline{q(z_n)}$. Such a change of coordinates preserves (3.11) and the conditions $\operatorname{tr} F_{22} = 0$ and $(\operatorname{tr})^2 F_{32} = 0$. If q additionally satisfies a certain third-order ODE, then the condition $(\operatorname{tr})^3 F_{33} = 0$ also holds. Since $q(0) = 0$, the solution to this equation is uniquely determined by its first- and second-order derivatives at the origin.

Thus, the normalization of Γ depends on the following data:

(i) the point $p \in \Gamma$ corresponding to the origin in the normal form;

(ii) a noncomplex tangential direction at p defining the chain γ which has the equation $'z = 0$, $y_n = 0$ in the normal form;

(iii) the choice of an orthonormal basis in $H_p\Gamma$;

(iv) two real parameters fixing the parametrization of γ.

The main result of Moser's theory can be stated as follows.

Theorem 3.2. *For each choice of the initial data* (i)–(iv) *there exists a unique biholomorphic mapping h taking Γ to the normal form.*

The normal form of the sphere is $y_n = |'z|^2$. Every choice of the initial conditions (i)–(iv) determines a unique linear fractional automorphism of the sphere. Therefore, by Moser's theorem every local automorphism is global. We obtain the Poincaré–Alexander theorem (see the next section for a detailed discussion).

Many applications of Moser's theory are contained in Vitushkin's expository papers [152, 153].

3.3. Equivalence problem: the Cartan–Chern approach. Cartan's equivalence problem (in the local form) can be stated as follows. Let U and V be open subsets in $\mathbb{R}^n$. Suppose that $\omega_U = (\omega_U^1, \dots, \omega_U^n)$ and $\Omega_V = (\Omega_V^1, \dots, \Omega_V^n)$ are co-frames (bases of 1-forms) on U and V, respectively. Consider a prescribed linear group G. The problem is to determine all diffeomorphisms

$$f\colon U \to V$$

satisfying the condition

$$f^*\Omega_V = \omega_U \gamma_{UV} \qquad \text{with} \quad \gamma_{UV} \in G.$$

In the original work of É. Cartan, the group G is allowed to vary from point to point. Note that many natural equivalence problems in differential geometry (for Riemannian structures, differential equations, CR structures, etc.) can be represented in this form for an appropriate choice of the co-frames and the group G.

The approach of É. Cartan to the solution of the above equivalence problem is based on the observation that the problem has a (relatively) simple solution in the case of the trivial group $G = \{e\}$. Even then a complete solution of the equivalence problem is not quite explicit. In most cases this method provides a number of geometric invariants of the problem at hand (a common application of these invariants is to show that the equivalence problem does not admit any solution). The equivalence problem is considered to be solved if it is reduced to the problem with $G = \{e\}$. The main idea of this reduction consists in the introduction of a (finite) sequence of larger spaces reducing the group G at every step.

The procedure proposed by É. Cartan comprises several iterated steps. First, the co-frames ω and Ω must be extended in an equivariant way to $U \times G$ and $V \times G$. There, the first structure equations for $d\omega$ and $d\Omega$ can be written. These equations contain the so-called torsion terms, and the special procedure of absorption of these terms allows one to reduce the group. Note that this requires the expansion of the initial system to a higher bundle.

In the case of Levi nondegenerate hypersurfaces in $\mathbb{C}^n$ (not necessarily real analytic but sufficiently smooth) this approach leads to a complete solution of the local equivalence problem. This was achieved by É. Cartan [37] for $n = 2$ and by Chern [45] and Tanaka [145] in all dimensions. The adjacent equivalence problem for systems (3.9) was solved by Hachtroudi [93] using Cartan's method. This problem can also be viewed in terms of Cartan's theory of projective connections. This approach to the Segre geometry is developed by Chern [44] and Burns and Shnider [36].

Each of the methods described in this section gives useful information concerning local properties of biholomorphic mappings between strictly pseudoconvex or Levi nondegenerate hypersurfaces: precise bounds on the dimension of the automorphism group, efficient parametrization of biholomorphisms by their second-order jets, etc. Notice that the Moser and Cartan–Chern methods admit some generalizations to a wider class of real hypersurfaces (with special Levi degeneracies) or to real manifolds of higher codimension. However, in these cases the approach based on Segre varieties often turns out to be the most convenient and flexible. For example, the approach based on the dynamical description of the Segre family by an analytic ODE was recently extended to a wider class of hypersurfaces (with the degenerate Levi form) by Kossovskiy and Shafikov [103, 104]. Using the technique of the local theory of analytic dynamical systems and meromorphic differential equations, they studied the geometry of Levi degenerate hypersurfaces and their formal (i.e., given by formal power series) and analytic CR transformations. In particular, they proved in [104] that the formal and holomorphic equivalences of real analytic hypersurfaces in $\mathbb{C}^n$ do not coincide (for Levi nondegenerate hypersurfaces they coincide by Moser's theory). This is a consequence of the classical phenomenon of the local theory of analytic dynamical systems, where the formal and analytic classifications are different. Further results in this direction were obtained recently by Kossovskiy and Lamel [101].

4. HOLOMORPHIC MAPPINGS OF STRICTLY PSEUDOCONVEX DOMAINS AND SCALING

We begin with the special case of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$, $n > 1$, and the Poincaré–Alexander rigidity phenomenon [2, 131]: let $U, V \subset \mathbb{C}^n$ be neighbourhoods of points $p, q \in \partial\mathbb{B}^n$, respectively, and $f\colon U \to V$, $f(p) = q$, be a biholomorphic (or even nonconstant holomorphic) map which takes $U \cap \partial\mathbb{B}^n$ to $V \cap \partial\mathbb{B}^n$; then f extends to an automorphism of $\mathbb{B}^n$.

This result was used by Alexander [3] and Rudin [134] to prove the following

Theorem 4.1. *Any proper holomorphic self-map of the unit ball $\mathbb{B}^n$, $n > 1$, is an automorphism of $\mathbb{B}^n$.*

The Poincaré–Alexander phenomenon and Theorem 4.1 illustrate a great difference between biholomorphic and proper holomorphic mappings in one and several complex variables. The following result obtained by C. Fefferman [73] has been influential for the development of the theory of holomorphic mappings.

Theorem 4.2. *Let $f\colon \Omega \to \Omega'$ be a biholomorphic mapping between two strictly pseudoconvex domains in $\mathbb{C}^n$ with C^∞-smooth boundaries. Then f extends to $\overline{\Omega}$ as a mapping of class C^∞.*

The original proof of this theorem was long and complicated, but it stimulated intensive research on the boundary regularity of proper holomorphic mappings, which has led to the discovery of new methods and results. In this section we present the main steps of a different, more elementary, approach to the proof of Theorem 4.2.

4.1. The Schwarz reflection principle. The following theorem may be considered as the Schwarz reflection principle in $\mathbb{C}^n$, $n > 1$.

Theorem 4.3. *Let Ω be a one-sided neighbourhood of a strictly pseudoconvex real analytic hypersurface Γ in $\mathbb{C}^n$. Let also Γ' be a real analytic strictly pseudoconvex hypersurface in $\mathbb{C}^n$. Suppose that $f\colon \Omega \to \mathbb{C}^n$ is a holomorphic mapping of class $C^1(\Omega \cup \Gamma)$ such that $f(\Gamma) \subset \Gamma'$. Then f extends holomorphically to a full neighbourhood of Γ in $\mathbb{C}^n$.*

Corollary 4.4. *Let Ω and Ω' be strictly pseudoconvex domains with real analytic boundaries and $f\colon \Omega \to \Omega'$ be a proper holomorphic map which extends to $\overline{\Omega}$ as a C^1 map. Then f extends holomorphically to a neighbourhood of $\overline{\Omega}$.*

We present here two different proofs of this theorem, both of which are local. The first one was obtained in [117] and [110].

Fix a point $p \in \Gamma$. Let ρ and ψ be strictly plurisubharmonic real analytic local defining functions of Γ and Γ' near p and $f(p)$, respectively. As usual, we consider them as power series in z and $\overline{z}$. Then

$$\psi(f(z), \overline{f(z)}) = 0 \tag{4.1}$$

whenever

$$\rho(z, \overline{z}) = 0. \tag{4.2}$$

Let $X_1, \dots, X_{n-1}$ be a basis in the space of tangential Cauchy–Riemann operators on Γ. Applying them to (4.1), via the chain rule we obtain

$$X_j \psi(f(z), \overline{f(z)}) = \sum_{k=1}^{n} \frac{\partial \psi}{\partial w_k}(f(z), \overline{f(z)}) X_j f(z) = 0, \qquad j = 1, \dots, n-1. \tag{4.3}$$

It suffices to consider the case when f is not constant. Since both hypersurfaces are strictly pseudoconvex, we conclude that the tangent mapping $df(p)$ is nondegenerate. Together with the nondegeneracy of the Levi form of ψ at $f(p)$, this allows us to apply the implicit function theorem to system (4.1), (4.3). We obtain

$$\overline{f(z)} = H\bigl(f(z), X_1 f(z), \dots, X_{n-1} f(z)\bigr), \qquad z \in \Gamma, \tag{4.4}$$

where H is a holomorphic function in all variables. Let $\mathbb{C} \ni \zeta \mapsto l_c(\zeta)$ be a family of parallel complex affine lines depending on a parameter $c \in \mathbb{C}^{n-1}$ that are transverse to Γ near p. Then the intersection of every line with Γ is a real analytic curve γ_c in $\mathbb{C}$. Every coefficient of X_j is a real analytic function, and its restriction to γ_c extends to a function holomorphic in a neighbourhood (whose size is independent of c) of γ_c. Replacing in (4.4) these coefficients with such holomorphic extensions, by the one-dimensional Schwarz reflection principle we conclude that the restriction of f to every linear section $\Omega \cap l_c$ extends holomorphically across Γ. Then the result follows by the Hartogs lemma.

The second proof, due to Webster [157], is shorter. Since the hypersurfaces are strictly pseudoconvex, the projectivizations of their holomorphic tangent bundles $\mathbb{P}H(\Gamma)$ and $\mathbb{P}H(\Gamma')$ are totally real submanifolds of maximal dimension $2n-1$. Then the mapping $z \mapsto (z, df(z))$ is holomorphic on a wedge-type domain with the edge $\mathbb{P}H(\Gamma)$, is continuous up to the edge, and takes it to $\mathbb{P}H(\Gamma')$. Hence, this mapping extends holomorphically to a neighbourhood of the edge by the reflection principle (see Proposition 2.10).

These two proofs of Theorem 4.3 represent two different types of the reflection principle: analytic and geometric. They both require additional assumptions on the regularity of f on the hypersurface Γ. The analytic approach needs C^1 regularity, while the geometric one requires slightly weaker regularity, namely, continuity of the lift of f up to $\mathbb{P}H(\partial\Omega)$. Furthermore, the geometric reflection principle admits smooth generalizations. A C^∞-smooth version of Theorem 4.3 was obtained by Nirenberg, Webster, and Yang [116]: if the hypersurfaces Γ and Γ' are merely C^∞-smooth, then f is necessarily of class $C^\infty(\Omega \cup \Gamma)$. The proof follows along the same lines with the application of Proposition 2.11.

4.2. Continuous extension. We need the following version of the classical Hopf lemma.

Proposition 4.5. *Let Ω be a bounded domain with C^2-smooth boundary in $\mathbb{C}^n$ and let K be a compact subset of Ω. For every constant $L > 0$ there exists a constant $C = C(K, L) > 0$ with the following property: if a function $u \in \mathrm{PSH}(\Omega)$ is such that $u(z) < 0$ for every $z \in \Omega$ and $u(z) \le -L$ for all $z \in K$, then $|u(z)| \ge C \operatorname{dist}(z, \partial\Omega)$ for all $z \in \Omega$.*

One of the first results on the boundary behaviour of holomorphic mappings (see [94, 118]) is the following

Theorem 4.6. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping between two strictly pseudoconvex domains in $\mathbb{C}^n$. Then f extends to $\overline{\Omega}$ as a $1/2$-Hölder continuous mapping.*

Let ρ and ψ be strictly plurisubharmonic global defining functions of Ω and Ω', respectively. The functions $v(z) = \psi(f(z))$ and $u(p) = \sup\{\rho(z) : f(z) = p\}$ are plurisubharmonic in Ω and Ω', respectively. Applying the Hopf lemma to these functions, we obtain $C\rho(z) \le \psi(f(z)) \le C^{-1}\rho(z)$ for some constant $C > 0$. This is equivalent to the boundary distance preserving property

$$C \operatorname{dist}(z, \partial\Omega) \le \operatorname{dist}(f(z), \partial\Omega') \le C^{-1} \operatorname{dist}(z, \partial\Omega). \tag{4.5}$$

From the decreasing property of the Kobayashi–Royden pseudometric and the estimates for this metric from above and below, it follows that

$$C|df(z)V| \operatorname{dist}(f(z), \partial\Omega')^{-1/2} \le F_{\Omega'}(f(z), df(z)V) \le F_\Omega(z, V) \le C^{-1}|V| \operatorname{dist}(z, \partial\Omega)^{-1}$$

for every point $z \in \Omega$ and every tangent vector V. In view of (4.5) this implies the estimate

$$\|df(z)\| \le C \operatorname{dist}(z, \partial\Omega)^{-1/2} \tag{4.6}$$

for the operator norm of the differential. The theorem now follows by the classical integration argument of Hardy and Littlewood.

The same proof works with minor modifications when the domain Ω is merely pseudoconvex: instead of the defining function ρ one can use the bounded exhaustion function of Diederich and Fornaess (Theorem 2.2). The assumptions on f and Ω' can also be weakened (see Proposition 6.3 below). It was the idea of Diederich and Fornaess [61] to utilize the Kobayashi–Royden metric instead of the previously used Carathéodory metric.

Theorem 4.6 does not allow one to immediately deduce Fefferman's Theorem 4.2 from Theorem 4.3 and its smooth counterpart. However, it was used by Nirenberg, Webster, and Yang [116] to prove the continuity of the lift of f up to $\mathbb{P}H(\partial\Omega)$. This was done with rather tricky and subtle arguments involving the Julia–Carathéodory lemma. The argument was later simplified by Forstnerič [79]. In the next subsection we present a more transparent proof using scaling.

4.3. The scaling method. Let Ω be a domain with a strictly pseudoconvex boundary of class C^2 and a defining function ρ near a point $w^0 \in \partial\Omega$. There exists a neighbourhood U of w^0 in $\mathbb{C}^n$ and a family of biholomorphic mappings $h_w\colon \mathbb{C}^n \to \mathbb{C}^n$, depending continuously on $w \in \partial\Omega \cap U$, such that the following conditions are satisfied:

(i) $h_w(w) = 0$;

(ii) the defining function $\rho_w := \rho \circ h_w^{-1}$ for the domain $h_w(\Omega)$ has the form

$$\rho_w = 2\operatorname{Re} z_n + 2\operatorname{Re} Q_w(z) + H_w(z) + R_w(z),$$

where $R_w(z) = o(|z|^2)$, $Q_w(z) = \sum_{\mu,\nu=1}^n q_{\mu\nu}(w) z^\mu z^\nu$, and $H_w(z) = \sum_{\mu,\nu=1}^n h_{\mu,\nu}(w) z^\mu \overline{z}^\nu$; furthermore, $Q_w(z) = 0$ and $H_w(z) = 0$ when $z_n = 0$;

(iii) each mapping h_w sends the real normal of $\partial\Omega$ at the point w to the real normal $\{z_1 = \ldots = z_{n-1} = \operatorname{Im} z_n = 0\}$ of $\partial h_w(\Omega_w)$ at the origin.

In applications of this construction one can usually assume that $w^0 = 0$ and $\partial\Omega$ is already normalized near the origin; therefore, one can additionally assume that h_{w^0} is the identity mapping.

As before, $'z = (z_1, \ldots, z_{n-1})$ so that $z = ('z, z_{n-1})$. Consider a sequence of points $\{q^k\}$ in Ω that converges to a point $q \in \partial\Omega$. Denote by $w^k \in \partial\Omega$ the point closest to q^k. Set $h^k := h_{w^k}$ and $\rho_k := \rho_{w^k}$. Set $\delta_k = \operatorname{dist}(h^k(q^k), \partial h^k(\Omega))$. Then $h^k(q^k) = ('0, -\delta_k)$. Consider the dilations

$$d^k\colon \ ('z, z_n) \mapsto \bigl(\delta_k^{-1/2}\,'z, \delta_k^{-1} z_n\bigr).$$

Finally, define the biholomorphic mappings $D^k := d^k \circ h^k$. Note that this sequence of biholomorphic mappings is determined by Ω and the sequence $\{q^k\}$. We call the sequence $\{D^k\}$ the *scaling along a sequence* $\{q^k\}$. Let $\Omega_k = D^k(\Omega) = \{\delta_k^{-1}\rho_k \circ d_k^{-1} < 0\}$. It is easy to see that the sequence of functions $\{\delta_k^{-1}\rho_k \circ d_k^{-1}\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to the function $2\operatorname{Re} z_n + |'z|^2$, which defines the domain $\mathbb{H}$ given by (2.1). As a consequence, the sequence of domains $\{\Omega_k\}$ converges to $\mathbb{H}$ with respect to the Hausdorff distance.

Scaling along a sequence has many applications. As an example, we conclude the sketch of the proof of Fefferman's mapping theorem using the arguments from [127]. It suffices to show that under the hypothesis of the theorem the lift $(z, df(z))$ of f to the tangent bundle extends continuously to $\mathbb{P}H(\partial\Omega)$. Arguing by contradiction, assume that there exists a sequence of points $\{p^k\}$ in Ω converging to a boundary point p such that their images q^k converge to some point $q \in \partial\Omega'$, but the sequence $\{p^k, df(p^k)\}$ does not converge to $\mathbb{P}H(\partial\Omega')$. Let $\{G^k\}$ and $\{D^k\}$ be the scaling sequences along the sequences $\{p^k\}$ and $\{q^k\}$, respectively. Then one can show that the sequence $\{f^k = D^k \circ f \circ (G^k)^{-1}\}$ converges to a holomorphic mapping which is degenerate at some point. On the other hand, it is easy to see by the standard normal family argument that the limit map is a biholomorphism of the unit ball. This contradiction proves the theorem.

The idea of using almost holomorphic functions (i.e., functions with asymptotically vanishing $\overline{\partial}$-operator) and the reflection principle in Fefferman's theorem is due to Nirenberg, Webster, and Yang [116]. It was also used for hypersurfaces of class C^m with noninteger $m > 2$ in [127], where it was proved that a biholomorphic mapping $f\colon \Omega \to \Omega'$ between strictly pseudoconvex domains with boundaries of class C^m extends to $\overline{\Omega}$ as a map of class $C^{m-1}(\overline{\Omega})$. A similar result but by different methods was proved by Lempert [108]. Later Khurumov [99] proved that, in fact, $f \in C^{m-1/2}(\overline{\Omega})$. This result is sharp.

4.4. Proper and locally proper mappings. The case when $f\colon \Omega \to \Omega'$ is a proper mapping can be reduced to the biholomorphic case via the following generalization of Alexander's theorem obtained in [122].

Theorem 4.7. *Let $f\colon \Omega \to \Omega'$ be a proper holomorphic mapping between two strictly pseudoconvex domains with C^2-smooth boundaries. Then $df(z)$ is nondegenerate at every point $z \in \Omega$; i.e., f is locally biholomorphic.*

Arguing by contradiction, assume that the Jacobian determinant of f vanishes on a complex hypersurface H in Ω. Let $\{p^k\}$ be a sequence of points in H converging to a boundary point, and let $q^k = f(p^k)$. Consider the scalings $\{G^k\}$ and $\{D^k\}$ along these sequences. Then the sequence $\{f^k = D^k \circ f \circ (G^k)^{-1}\}$ converges to a proper holomorphic mapping $F\colon \mathbb{H} \to \mathbb{H}$, which is a biholomorphism by Alexander's theorem. On the other hand, it follows from the choice of $\{p^k\}$ that dF vanishes at some interior point, which is a contradiction.

Corollary 4.8. *A proper holomorphic self-mapping of a strictly pseudoconvex domain with C^2-smooth boundary is a biholomorphism.*

A similar idea allows one to establish the following rigidity phenomenon for CR mappings [130].

Theorem 4.9. *Let Γ and Γ' be strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$. Suppose that U is a neighbourhood of a point $p \in \Gamma$ and $f\colon \Gamma \cap U \to \Gamma'$ is a continuous nonconstant CR mapping. Then there exist neighbourhoods V and V' of p and $f(p)$, respectively, such that $f\colon \Gamma \cap V \to \Gamma' \cap V'$ is a homeomorphism.*

We can use the fact that f extends holomorphically to the pseudoconvex one-sided neighbourhood of p. The difficulty is that a priori f may not be a proper mapping there. The main idea of the proof is to show that the set $f^{-1}(f(p))$ is finite in a neighbourhood of p on $\partial\Omega$. This will imply that f is proper and will reduce the problem to the previous theorem. Arguing by contradiction, suppose that this set contains a sequence converging to p and apply again the scaling (see [55] for a more general scaling result needed here). Then one can show that the limit map is an automorphism of the ball and at the same time is degenerate at some point, which is a contradiction.

5. EXTENSION OF GERMS OF HOLOMORPHIC MAPPINGS

In this section we present the results which generalize and develop the rigidity phenomenon discovered by Poincaré and Alexander.

A real hypersurface $\Gamma = \rho^{-1}(0)$ in $\mathbb{C}^n$ is called *real algebraic* if it is defined by a real polynomial ρ. For an open set $U \subset \mathbb{C}^n$, a holomorphic map $F\colon U \to \mathbb{C}^n$ is called *algebraic* if its graph is contained in an algebraic subvariety of $\mathbb{C}^{2n}$ of dimension n. The following result is due to Webster [155].

Theorem 5.1. *Let Γ and Γ' be Levi nondegenerate real algebraic hypersurfaces in $\mathbb{C}^n$ of degrees m and m', respectively, and let U be a neighbourhood in $\mathbb{C}^n$ of a point $p \in \Gamma$. Suppose that $f\colon U \to \mathbb{C}^n$ is a holomorphic mapping with a nondegenerate differential at p and such that $f(\Gamma) \subset \Gamma'$. Then f extends to $\mathbb{C}^n$ as an algebraic mapping of degree bounded above by a constant depending only on n, m, and m'.*

Applying complex conjugation, we rewrite equation (4.4) in the form $f(z) = G(z, \overline{z}, \overline{F(z)})$, where $F = (f, J_f)$. Here G is a (holomorphic) algebraic function and J_f denotes the Jacobian matrix of f viewed as a $\mathbb{C}^{n^2}$-valued map. It follows that the mapping F takes Γ to the real algebraic set $M = \{(z,\zeta,\omega) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n^2}\colon \zeta = G(z,\overline{z},\overline{\omega})\}$. By Lemma 3.1, $F(Q_w)$ is contained in $Q'_{F(w)}$ (the Segre variety defined by M), which implies that the restriction of f to Q_w is an algebraic map (of controlled degree). Consider n families of transverse Segre varieties for Γ. After a local biholomorphic and algebraic change of coordinates one can transform them to families of parallel coordinate hyperplanes. Now the classical theorem on separate algebraicity [32] can be applied (see [138] for details).

We begin the discussion of the analytic case with the result of [120].

Theorem 5.2. *Let Γ be a (connected) real analytic strictly pseudoconvex hypersurface in $\mathbb{C}^n$, $n > 1$, U a neighbourhood of a point $p \in \Gamma$, and $f\colon U \to \mathbb{C}^n$ a nonconstant holomorphic mapping,* *and assume that $f(U \cap \Gamma) \subset \partial\mathbb{B}^n$. Then f can be continued along any path on Γ starting at p as a locally biholomorphic mapping.*

Corollary 5.3. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, with real analytic simply connected boundary. Assume that f is a nonconstant holomorphic mapping in a neighbourhood U of a point $p \in \partial\Omega$ such that $f(U \cap \partial\Omega) \subset \partial\mathbb{B}^n$. Then f extends to a biholomorphic mapping between Ω and $\mathbb{B}^n$.*

The proof uses the reflection principle. In our case equation (4.1) has the form $(f, f) - 1 = 0$. Hence, in (4.3) one can apply the Cramer rule (instead of the implicit function theorem). It follows that H in (4.4) is a rational function of f and $X_j f$. This allows one to extend f along the family l_c of complex lines meromorphically but "far enough." With this the proof can be completed as follows. First we complexify $\partial\Omega$ near a given point p by a biholomorphic change of coordinates. Then we cut off a piece of $\partial\Omega$ by a real hyperplane parallel to the tangent plane at p. Next we extend f meromorphically along a family of complex lines parallel to this hyperplane. Repeating this procedure, we obtain a global meromorphic extension of f. The last step is to prove that this extension is, in fact, holomorphic.

A real analytic hypersurface Γ is called *spherical* at a point $p \in \Gamma$ if in a neighbourhood of p it is locally biholomorphic to an open piece of the real sphere $\partial\mathbb{B}^n$. It follows from Theorem 5.2 that if a connected Γ is spherical at one point, then it is spherical at every point. Burns and Shnider [34] constructed the following example. Let $\Gamma = \{z \in \mathbb{C}^2 \colon y_2 = |z_1|^2\}$ (unbounded sphere) and $\Gamma' = \{z \in \mathbb{C}^2 \colon \sin \ln|z_2|^2 = 0,\ e^{-\pi} \le |z_2| \le 1\}$. Then the mapping $f(z) = (z_1/\sqrt{z_2}, \exp\{i \ln z_2\})$ with a suitably chosen branch of $\ln z_2$ maps $\Gamma \setminus \{0\}$ into Γ' but does not extend even continuously to $z = 0$. In this example Γ' is a compact real analytic spherical hypersurface which is not simply connected.

A result similar to Theorem 5.2 holds if the sphere in the target space is replaced with an algebraic hypersurface (see [137]).

Theorem 5.4. *Let Γ be a connected essentially finite real analytic hypersurface in $\mathbb{C}^n$ and $p \in \Gamma$. Let Γ' be a compact strictly pseudoconvex real algebraic hypersurface in $\mathbb{C}^n$. Let f be a germ of a holomorphic mapping from Γ to Γ' defined at p. Then f extends holomorphically along any path on Γ with the extension sending Γ to Γ'.*

In this result the hypersurface Γ is not assumed to be strictly pseudoconvex, although it follows from the proof that it is pseudoconvex and that the set of weakly pseudoconvex points of Γ consists precisely of the points where the extended map degenerates. The proof is based on the technique of Segre varieties. The hypersurface Γ is called *essentially finite* if the map $z \to Q_z$ is locally finite near every point of Γ. The main idea is the holomorphic extension along Segre varieties: from the properties of Segre varieties (see Section 3) one can conclude that the inclusion $f(Q_z) \subset Q'_{f(z)}$ must hold not only for points in the domain of f but also for points z with the property that Q_z has a nonempty intersection with the open set where f is defined. This gives a holomorphic extension of f to such points, which can be quite far away from the initial domain of f, no matter how small it is. An iterative procedure then gives the extension of f along any curve in Γ. In particular, this technique can be used to give an alternative independent proof of Theorem 5.2. In [102], this technique was further refined to show that the germ of a biholomorphic mapping $f \colon \Gamma \to \Gamma'$ also extends across complex hypersurfaces that might be present in Γ. Here Γ' is either a sphere or, more generally, any nondegenerate hyperquadric in $\mathbb{C}^n$.

Consider now the case of a real analytic hypersurface in the target domain. It turns out that unlike the spherical case, for nonspherical strictly pseudoconvex hypersurfaces the phenomenon of analytic continuation holds without any additional topological restrictions. This is a consequence of the following result.

Theorem 5.5. *Let Γ and Γ' be nonspherical real analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$, $n > 1$, and U be a neighbourhood of a point $p \in \Gamma$, where the sets Γ, Γ', U, and $\Gamma \cap U$*

are connected and Γ' *is compact. Suppose that there exists a nonconstant holomorphic mapping* $f\colon U \to \mathbb{C}^n$ *such that* $f(U \cap \Gamma) \subset \Gamma'$. *Then* f *continues analytically along any path in* Γ *as a locally biholomorphic mapping.*

In this form this theorem was proved in [121]. The proof is based on a careful analysis of the behaviour of Moser's chains on a nonspherical strictly pseudoconvex hypersurface. Vitushkin, Ezhov, and Kruzhilin [154] obtained a different proof of this result in a more general setting when Γ and Γ' are nonspherical real analytic strictly pseudoconvex hypersurfaces in arbitrary n-dimensional complex manifolds, $n \geq 2$. We refer the reader to the surveys [152, 153] by Vitushkin for a comprehensive discussion and further results in this direction.

For nonalgebraic Γ', the problem of analytic continuation remains open in the presence of weakly pseudoconvex points in Γ; for example, it is not known whether the map f in Theorem 5.5 extends to a neighbourhood of an isolated weakly pseudoconvex point that Γ might have. The difficulty is that the results of Moser's theory do not hold in general near points where the Levi form of Γ degenerates, and it is also not clear how to generalize the technique of analytic continuation along Segre varieties for nonalgebraic target hypersurfaces.

We conclude this section with a result by Nemirovski and Shafikov [115] on uniformization of strictly pseudoconvex domains.

Theorem 5.6. *Let* Ω *and* Ω' *be strictly pseudoconvex domains with real analytic boundaries. Then the universal coverings of* Ω *and* Ω' *are biholomorphically equivalent if and only if the boundaries of these domains are locally biholomorphically equivalent.*

If the boundaries of Ω and Ω' are locally biholomorphically equivalent somewhere, then by Theorems 5.2 and 5.5 the germ of the equivalence map extends as a locally biholomorphic map f along any path in $\partial\Omega$ and, hence, along any path in a one-sided neighbourhood V of $\partial\Omega$. This possibly multiple-valued map on V extends to the envelope of holomorphy of V, which is Ω by Hartogs' theorem. Kerner's theorem [98] states that the envelope of holomorphy of the universal covering $\widehat{V}$ of V is the universal covering of the envelope of holomorphy of V. This implies that f extends to a map $f\colon \widehat{\Omega} \to \Omega'$. In the nonspherical case, the final result can be deduced by repeating the argument for the inverse of the equivalence of $\partial\Omega$ and $\partial\Omega'$. In the spherical case, an additional argument using invariant metrics in needed. The proof of Theorem 5.6 in the other direction essentially follows the general scheme outlined in Section 4: the equivalence map $f\colon \widehat{\Omega} \to \widehat{\Omega}'$ is first extended smoothly to the boundary, and then the reflections principle (Theorem 4.3) is applied.

6. WEAKLY PSEUDOCONVEX DOMAINS

6.1. Finite type and plurisubharmonic peak functions. Consider a smooth real hypersurface $\Gamma = \rho^{-1}(0)$ in $\mathbb{C}^n$. The following notion of the type of Γ at a point $p \in \Gamma$ is due to D'Angelo [57]. Denote by O_p the space of germs of holomorphic mappings $h\colon (\mathbb{C}, 0) \to \mathbb{C}^n$, $h(0) = p$. Denote by $\nu(h)$ the order of vanishing of $h - h(0)$ at the origin. Let also $\nu(\rho \circ h)$ denote the order of vanishing of the function $\rho \circ h$ at the origin. Then the type of Γ at 0 is defined as

$$\tau(\Gamma, p) = \sup\left\{ \frac{\nu(\rho \circ h)}{\nu(h)},\ h \in O_p \right\}. \tag{6.1}$$

In general the function $p \mapsto \tau(\Gamma, p)$ is not upper semicontinuous. Nevertheless, D'Angelo proved the following. Let Ω be a smoothly bounded pseudoconvex domain and let $q \in \partial\Omega$. Then there exists a neighbourhood U of q such that for each $p \in U \cap \partial\Omega$

$$\tau(\partial\Omega, p) \leq \frac{\tau(\partial\Omega, q)^{n-1}}{2^{n-2}}. \tag{6.2}$$

A real analytic hypersurface is of finite type if and only if it contains no germs of complex analytic sets of positive dimension. It is a result of Diederich and Fornaess [60] that a compact real analytic subset of $\mathbb{C}^n$ contains no nontrivial complex analytic subsets, and so every bounded domain with real analytic boundary in $\mathbb{C}^n$ is of finite type at every boundary point.

The following characterization of finite type is due to Fornaess and Sibony [75, 141].

Proposition 6.1. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ and $p \in \partial\Omega$. Assume there exist a function $\phi_p \in C^0(\overline{\Omega})$ plurisubharmonic in Ω and constants $C > 0$, $\lambda > 0$, and $k > 0$ such that*

$$-C|z-p|^\lambda \le \phi_p(z) \le -|z|^{2k\lambda}, \qquad z \in \overline{\Omega}. \tag{6.3}$$

Then $\partial\Omega$ is of type less than $2k$ at p.

The function ϕ_p above is called a *plurisubharmonic barrier* (at p). If $k = 1$, the existence of a barrier is equivalent to strict pseudoconvexity at p (see [140]).

We say that a boundary point $q \in \partial\Omega$ satisfies the *barrier property* if there exists a neighbourhood U of q such that every point $p \in U \cap \partial\Omega$ admits a barrier function (with k and λ independent of p).

Theorem 6.2. *Let Ω be a domain in $\mathbb{C}^n$ with a smooth pseudoconvex boundary of finite type in a neighbourhood V of a point $q \in \partial\Omega$. Then q satisfies the barrier property.*

When $n = 2$ or when $V \cap \partial\Omega$ is convex, this result is due to Fornaess and Sibony [75]. The real analytic case is due to Diederich and Fornaess [63]. Finally, the general case was treated by Cho [50].

Diederich and Fornaess [61] proposed the use of barrier functions in order to obtain lower bounds for the Kobayashi–Royden metric. This approach works for a wide class of domains. As an application we present the following result obtained in [143].

Proposition 6.3. *Let Ω and Ω' be domains in $\mathbb{C}^n$ whose boundaries are C^2-smooth near some points $p \in \partial\Omega$ and $p' \in \partial\Omega'$ that satisfy the barrier property. Let $f\colon \Omega \to \Omega'$ be a holomorphic mapping such that for some neighbourhood U of p the cluster set $C_\Omega(f; U \cap \partial\Omega)$ does not intersect Ω'. Assume also that the cluster set $C_\Omega(f;p)$ contains the point p'. Then f extends to a neighbourhood of p in $\partial\Omega$ as a Hölder continuous mapping.*

Various results concerning continuous extension of holomorphic mappings are also obtained in [25, 26, 61, 74, 85].

6.2. $\overline{\partial}$-Approach to boundary regularity. Let Ω be a bounded domain in $\mathbb{C}^n$. Consider the Hermitian Hilbert space $L^2(\Omega)$ equipped with the standard Hermitian product $(\cdot,\cdot)_{L^2(\Omega)}$. Then $\mathcal{O}(\Omega) \cap L^2(\Omega)$ is a closed subspace in $L^2(\Omega)$ and hence is itself a Hilbert space. Fix a point $p \in \Omega$. The evaluation map

$$l_p\colon\ \mathcal{O}(\Omega) \cap L^2(\Omega) \to \mathbb{C}, \qquad h \mapsto h(p)$$

is a bounded linear functional on $\mathcal{O}(\Omega) \cap L^2(\Omega)$. By the Riesz representation theorem there exists a unique element in $\mathcal{O}(\Omega) \cap L^2(\Omega)$, which is denoted by $K_\Omega(\cdot,p)$, such that

$$h(p) = l_p(h) = (h, K_\Omega(\cdot,p))_{L^2(\Omega)}$$

for all $h \in \mathcal{O}(\Omega) \cap L^2(\Omega)$. The function $K_\Omega\colon \Omega \times \Omega \to \mathbb{C}$ is called the Bergman kernel for Ω. By this definition the function $z \mapsto K_\Omega(z,p)$ is in $L^2(\Omega)$ for every $p \in \Omega$. Furthermore, $K_\Omega(z,p) = \overline{K_\Omega(p,z)}$ and the function $(z,w) \mapsto K_\Omega(z,\overline{w})$ is holomorphic on $\Omega \times \Omega$. The orthogonal projection operator

$$P_\Omega\colon\ L^2(\Omega) \to \mathcal{O}(\Omega) \cap L^2(\Omega)$$

is called the Bergman projection. One has $P_\Omega(h) = (h, K_\Omega)_{L^2(\Omega)}$. The following transformation rules (see, for example, [132]) play a key role in the application of the Bergman kernel and the Bergman projection to holomorphic mappings.

Theorem 6.4. *Let $f\colon \Omega_1 \to \Omega_2$ be a biholomorphic mapping between bounded domains in $\mathbb{C}^n$. Then*

$$K_{\Omega_1}(p,z) = J_f(p)K_{\Omega_2}(f(p), f(z))\overline{J_f(z)}, \qquad P_{\Omega_1}(J_f h \circ f) = J_f(P_{\Omega_2}(h) \circ f)$$

for all $h \in L^2(\Omega_2)$. Here J_f denotes the determinant of the Jacobian matrix of f.

The above transformation rule for the Bergman projection (but not for the kernel) also remains true for proper holomorphic mappings.

A smoothly bounded pseudoconvex domain Ω is said to satisfy *Condition* R if

$$P_\Omega(C^\infty(\overline{\Omega})) \subset C^\infty(\overline{\Omega}).$$

The following result is due to Bell and Catlin [23] and to Diederich and Fornaess [62].

Theorem 6.5. *Let $f\colon \Omega_1 \to \Omega_2$ be a proper holomorphic mapping between smoothly bounded pseudoconvex domains. Suppose that Ω_1 satisfies Condition* R. *Then f extends as a C^∞-smooth mapping to $\overline{\Omega}_1$.*

A version of this theorem for CR mappings between the boundaries of domains was obtained by Bell and Catlin [24].

A general approach to verify Condition R for a prescribed class of domains relies on the $\overline{\partial}$-Neumann problem. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let g be a $\overline{\partial}$-closed $(0,1)$-differential form with coefficients of class $L^2(\Omega)$, i.e., $g \in L^2_{(0,1)}(\Omega)$. The $\overline{\partial}$-Neumann problem consists in determining the regularity of the solution u to the equation $\overline{\partial}u = g$ which is orthogonal to the kernel of the operator $\overline{\partial}$, that is, to the class $\mathcal{O}(\Omega) \cap L^2(\Omega)$. This solution is called the canonical solution. The operator

$$N_\Omega\colon L^2_{(0,1)}(\Omega) \to L^2(\Omega), \qquad g \mapsto u$$

is called the $\overline{\partial}$-Neumann operator on Ω. The relation between the Bergman projection and the $\overline{\partial}$-Neumann operator is given by Kohn's formula [100]

$$P_\Omega = \mathrm{Id} - \overline{\partial}^* N_\Omega \overline{\partial}.$$

Thus, if the $\overline{\partial}$-Neumann operator is globally regular, i.e., maps the space $C^\infty(\overline{\Omega})$ to itself, then Condition R holds. Regularity of the $\overline{\partial}$-Neumann operator has been an active area of research and led to the development of many important technical tools (see Catlin [38–40]). In particular, it is known that the existence of plurisubharmonic barriers (6.3) implies the regularity of the $\overline{\partial}$-problem (see [41, 42, 141]).

A smoothly bounded domain $\Omega \subset \mathbb{C}^n$ admits a defining function which is plurisubharmonic along the boundary if there exists a smooth defining function of Ω whose Levi form is positive semi-definite for all vectors at each boundary point (this condition is stronger than the pseudoconvexity, which requires that the Levi form is positive semi-definite on the holomorphic tangent space). Boas and Straube [31] proved that if Ω admits a defining function which is plurisubharmonic along the boundary, then it satisfies Condition R. In particular, every smoothly bounded convex domain satisfies Condition R.

Finally, it was shown by Christ [51] that for the worm domain of Diederich and Fornaess [58], which is smooth and pseudoconvex, Condition R does not hold. This shows limitations of the approach and raises an important question of finding sufficient and necessary conditions for the regularity of the Bergman projection.

7. AUTOMORPHISM GROUPS AND PROPER SELF-MAPPINGS

The geometry of the boundary of a domain Ω in $\mathbb{C}^n$ influences the structure of the group of biholomorphic automorphisms of Ω. In turn, the automorphism group $\mathrm{Aut}(\Omega)$ may completely characterize the domain Ω. In this section we discuss some results in this direction.

7.1. Automorphisms of strictly pseudoconvex domains. We begin with the following result, generally known in the literature as the Wong–Rosay theorem.

Theorem 7.1. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume that $\mathrm{Aut}(\Omega)$ is not compact. Then Ω is biholomorphic to the unit ball $\mathbb{B}$.*

Since $\mathrm{Aut}(\Omega)$ is not compact, there exists a sequence $\{f^k\}$ in $\mathrm{Aut}(\Omega)$ which converges uniformly on every compact subset of Ω to a boundary point $q \in \partial\Omega$. Fix a point $p \in \Omega$ and set $q^k = f^k(p)$. Let $\{D^k\}$ be the scaling sequence for $\{q^k\}$. Then the sequence $F^k = D^k \circ f^k$ converges to a biholomorphic mapping from Ω to $\mathbb{B}$.

Theorem 7.1 was established by Webster [156] under the additional assumption that the group $\mathrm{Aut}(\Omega)$ has positive dimension. In full generality this result was obtained by Burns and Shnider [35] using the Chern–Moser theory. A more elementary approach based on invariant metrics is due to Wong [160] and Rosay [133]. The short proof presented above was given in [124, 126].

There are several other local versions of this result; here is one of them [72].

Theorem 7.2. *Let Ω be a domain (not necessarily bounded) in $\mathbb{C}^n$ and let $\partial\Omega$ be C^2-smooth and strictly pseudoconvex in a neighbourhood of a point $q \in \partial\Omega$. Suppose that there exists a sequence $\{f^k\}$ in $\mathrm{Aut}(\Omega)$ and a point $p \in \Omega$ such that $f^k(p) \to q$ as $k \to \infty$. Then Ω is biholomorphic to the unit ball $\mathbb{B}^n$.*

It is also interesting to consider the converse question: which groups can be realized as the automorphism group of a domain in $\mathbb{C}^n$? The following results are due to Winkelman [158, 159].

Theorem 7.3. *Let G be a (finite or infinite) countable group. Then there exists a (connected) Riemann surface M such that G is isomorphic to $\mathrm{Aut}(M)$.*

Theorem 7.4. *Let G be a connected (real) Lie group. Then there exists a Stein complete hyperbolic complex manifold M on which G acts effectively, freely, properly, and with totally real orbits, such that G is isomorphic to $\mathrm{Aut}(M)$.*

Note that any compact real Lie group can be realized as the automorphism group of a strictly pseudoconvex domain [16, 135].

7.2. Domains with large automorphism groups. Investigation of weakly pseudoconvex domains with large automorphism groups was initiated by Greene and Krantz [88]. The following result is due to Bedford and Pinchuk [19, 22].

Theorem 7.5. *Let Ω be a bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^2$. Suppose that $\mathrm{Aut}(\Omega)$ is not compact. Then Ω is biholomorphic to a domain of the form*

$$\{z \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^{2m} < 1\} \tag{7.1}$$

for some positive integer m.

The same result also holds if Ω is a smoothly bounded pseudoconvex domain of finite type. Furthermore, the assumption of pseudoconvexity (if the boundary is real analytic) can be dropped (see [20]).

We outline the proof of Theorem 7.5. Since $\mathrm{Aut}(\Omega)$ is noncompact, there exists a point $a \in \Omega$ and a sequence of automorphisms $\{f^j\}$ such that the sequence $q^k = f^k(a)$ converges to a boundary point $q \in \partial\Omega$. Then the sequence $\{f^k\}$ converges uniformly on compact subsets of Ω to a constant map $f^0 \equiv q$. Applying the scaling along the sequence $\{q^k\}$, one can prove that Ω is equivalent to a domain of the form $D = \{2x_2 + P(z_1, \overline{z}_1) = 0\}$, where P is a nonzero real polynomial. Note that the proof is more delicate than in the strictly pseudoconvex case and is based on precise estimates for the Kobayashi–Royden metric in pseudoconvex domains of finite type in $\mathbb{C}^2$. These estimates were obtained by Catlin [43]; a geometric proof of his result based on the scaling method was given by Berteloot [28]. The one-parameter group $L^t(z_1, z_2) = (z_1, z_2 + it)$ acts on the domain D. The biholomorphism $f\colon D \to \Omega$ defines a real one-parameter group of automorphisms $h^t = f \circ L^t \circ f^{-1}$.

The second step is to prove that the group $\{h^t\}$ is parabolic, that is, there exists a point $p \in \Omega$ (called a parabolic point) such that

$$\lim_{t\to-\infty} h^t(z) = \lim_{t\to\infty} h^t(z) = p.$$

The proof also uses the estimates for the Kobayashi metric.

The next step is to study the holomorphic vector field $X = (X_1, X_2)$ generating the parabolic subgroup $\{h^t\}$. This vector field is tangent to $\partial\Omega$, that is,

$$\operatorname{Re}\biggl(\frac{\partial\rho}{\partial z_1}X_1 + \frac{\partial\rho}{\partial z_2}X_2\biggr) = 0.$$

This condition leads to a rather precise description of the jet of $\partial\Omega$ at a parabolic point p, and this can be used to conclude the proof.

In the local case Verma [151] obtained the following classification result.

Theorem 7.6. *Let Ω be a bounded domain in $\mathbb{C}^2$. Suppose that there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \in \operatorname{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ converges to $p_\infty \in \partial\Omega$. Assume that the boundary of Ω is real analytic and of finite type near p_∞. Then exactly one of the following cases holds:*

(i) *if* $\dim \operatorname{Aut}(\Omega) = 2$, *then either*

   (a) Ω *is biholomorphic to* $\Omega_1 = \{z \in \mathbb{C}^2\colon 2\operatorname{Re} z_2 + P_1(\operatorname{Re} z_1) < 0\}$ *where* $P_1(\operatorname{Re} z_1)$ *is a polynomial that depends on* $\operatorname{Re} z_1$, *or*

   (b) Ω *is biholomorphic to* $\Omega_2 = \{z \in \mathbb{C}^2\colon 2\operatorname{Re} z_2 + P_2(|z_1|^2) < 0\}$ *where* $P_2(|z_1|^2)$ *is a homogeneous polynomial that depends on* $|z_1|^2$, *or*

   (c) Ω *is biholomorphic to* $\Omega_3 = \{z \in \mathbb{C}^2\colon 2\operatorname{Re} z_2 + P_{2m}(z_1, \overline{z}_1) < 0\}$ *where* $P_{2m}(z_1, \overline{z}_1)$ *is a homogeneous polynomial of degree* $2m$ *without harmonic terms*;

(ii) *if* $\dim \operatorname{Aut}(\Omega) = 3$, *then* Ω *is biholomorphic to* $\Omega_4 = \{z \in \mathbb{C}^2\colon 2\operatorname{Re} z_2 + (\operatorname{Re} z_1)^{2m} < 0\}$ *for some integer* $m \ge 2$;

(iii) *if* $\dim \operatorname{Aut}(\Omega) = 4$, *then* Ω *is biholomorphic to* $\Omega_5 = \{z \in \mathbb{C}^2\colon |z_1|^2 + |z_1|^{2m} < 0\}$ *for some integer* $m \ge 2$;

(iv) *if* $\dim \operatorname{Aut}(\Omega) = 8$, *then* Ω *is biholomorphic to* $\mathbb{B}^2$.

The dimensions 0, 1, 5, 6, *and* 7 *cannot occur with* Ω *as above.*

In higher dimensions the situation is more complicated. To the variables $z_1, \dots, z_n$ we assign the weights $\delta_1, \dots, \delta_n$, where $\delta_j = (2m_j)^{-1}$ for m_j a positive integer. If $J = (j_1, \dots, j_n)$ and $K = (k_1, \dots, k_n)$ are multi-indices, we set $\operatorname{wt}(J) = j_1\delta_1 + \ldots + j_n\delta_n$ and $\operatorname{wt}(z^J\overline{z}^K) = \operatorname{wt}(J) + \operatorname{wt}(K)$. We consider real polynomials of the form

$$p(z, \overline{z}) = \sum_{\operatorname{wt}(J)=\operatorname{wt}(K)=1/2} a_{JK} z^J \overline{z}^K. \tag{7.2}$$

The reality of p is equivalent to $a_{JK} = \overline{a_{KJ}}$. The balance of the weights $\operatorname{wt}(J) = \operatorname{wt}(K)$ implies that the domain

$$G = \bigl\{(w, z_1, \dots, z_n) \in \mathbb{C} \times \mathbb{C}^n\colon\ |w|^2 + p(z, \overline{z}) < 1\bigr\} \tag{7.3}$$

is invariant under the action of the real torus

$$(\phi, \theta) \mapsto \big(e^{i\phi} w, e^{i\delta_1 \theta} z_1, \dots, e^{i\delta_n \theta} z_n\big). \tag{7.4}$$

The weighted homogeneity of p implies that the Cayley-type transform $(w, z) \mapsto (w^*, z^*)$ defined by

$$w = \left(1 - \frac{iw^*}{4}\right)\left(1 + \frac{iw^*}{4}\right)^{-1}, \qquad z_j = z_j^* \left(1 + \frac{iw^*}{4}\right)^{-2\delta_j} \tag{7.5}$$

maps G biholomorphically onto the domain

$$D = \big\{(w, z_1, \dots, z_n) \in \mathbb{C} \times \mathbb{C}^n \colon \operatorname{Im} w + p(z, \overline{z}) < 0\big\}. \tag{7.6}$$

The latter is an unbounded realization of G. Note that D is invariant under the translation along the $\operatorname{Re} w$-direction. Since p is homogeneous, the domain D is invariant with respect to the family of anisotropic dilations. Hence the dimension of $\operatorname{Aut}(D)$ is at least 4.

Theorem 7.7. *Let $\Omega \Subset \mathbb{C}^{n+1}$ be a smoothly bounded convex domain of finite type. If* $\operatorname{Aut}(\Omega)$ *is noncompact, then Ω is equivalent to a domain of the form* (7.3).

This result is obtained in [21]. The scaling method in a convex domain Ω (not necessarily of finite type) relies on the estimates for the Kobayashi–Royden metric, which also have other applications. Denote by $L(a, V)$ the complex line passing through a point $a \in \Omega$ in the direction of a vector V. Define $\delta(a, V)$ to be the Euclidean distance from a to $L(a, V) \cap \partial\Omega$. Then the following estimate holds [21, 86]:

$$\frac{|V|}{2\delta(a, V)} \le F_\Omega(a, V) \le \frac{|V|}{\delta(a, V)}. \tag{7.7}$$

Using this estimate and his version of the scaling method, Frankel [86] proved the following

Theorem 7.8. *Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded convex domain and that there exists a discrete subgroup of* $\operatorname{Aut}(\Omega)$ *which acts properly discontinuously, freely, and cocompactly on Ω. Then Ω is a bounded symmetric domain.*

Recently, Zimmer [163] proposed a new approach to the problem of classifying convex domains with large automorphism groups. Let Ω be a domain in $\mathbb{C}^n$. The *limit set* of Ω is the set of points $z \in \partial\Omega$ for which there exists some $p \in \Omega$ and some sequence $\phi_k \in \operatorname{Aut}(\Omega)$ such that $\phi_k(p) \to z$. If $\operatorname{Aut}(\Omega)$ is noncompact, the limit set is not empty. If Ω is a bounded convex domain with C^1-smooth boundary, the *closed complex face* of a point $z \in \partial\Omega$ is the closed set $\partial\Omega \cap H_z(\partial\Omega)$. The main result of [163] is the following

Theorem 7.9. *Suppose Ω is a bounded convex domain with C^∞-smooth boundary. Then the following conditions are equivalent*:

(1) *the limit set of Ω intersects at least two closed complex faces of $\partial\Omega$*;

(2) *Ω is biholomorphic to the domain* (7.3).

Notice that there is no finite type assumption in this theorem. The main new tool used by Zimmer is the theory of Gromov hyperbolic metric spaces.

Suppose that (X, d) is a metric space. Let $I \subset \mathbb{R}$ be an interval. A curve $\sigma \colon I \to X$ is called *geodesic* if $d(\sigma(t_1), \sigma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in I$. A *geodesic triangle* is a choice of three points in X and geodesic segments connecting these points. A geodesic triangle is said to be *δ-thin* if any point on any of the sides of the triangle is within distance δ from the other two sides. A proper geodesic metric space (X, d) is called *δ-hyperbolic* if every geodesic triangle is δ-thin. If (X, d) is δ-hyperbolic for some $\delta \ge 0$, then (X, d) is called *Gromov hyperbolic.* Zimmer's approach uses the following result established in [162].

Theorem 7.10. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with smooth boundary. Then the following conditions are equivalent*:

(1) *Ω has a finite type in the sense of D'Angelo*;

(2) *(Ω, d_Ω) is Gromov hyperbolic, where d_Ω is the Kobayashi distance on Ω.*

Balogh and Bonk [7] established Gromov hyperbolicity of strictly pseudoconvex domains by using estimates for the Kobayashi–Royden metric (see Proposition 2.5). Further results in this direction were obtained recently by Bracci and Gaussier [33].

The condition of convexity is crucial in the proofs of the above results. It is not known whether any bounded pseudoconvex domain with smooth boundary of finite type and noncompact automorphisms group in $\mathbb{C}^n$, $n > 2$, is equivalent to (7.3). The question remains open even for domains with real algebraic boundaries.

7.3. Proper self-mappings. Finally, we discuss some progress in the direction originated from Alexander's Theorem 4.1: a proper holomorphic self-map of $\mathbb{B}^n$, $n > 1$, is a biholomorphism. Its generalization to the case of strictly pseudoconvex domains (see Corollary 4.8) is based on Theorem 4.7. However, the condition of strict pseudoconvexity is crucial for Theorem 4.7. Indeed, the map $f\colon (z_1, z_2) \to (z_1^2, z_2)$ takes the domain $\{z \in \mathbb{C}^2 \colon |z_1|^4 + |z_2|^2\}$ properly onto $\mathbb{B}^2$ but its critical locus is not empty. This is a serious obstacle for applications of the scaling method, and for this reason the analogs of Alexander's theorem are currently established only for special classes of domains.

One of the most general results in this direction was obtained by Bedford [14].

Theorem 7.11. *Let Ω be a bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$, $n \geq 2$. Then every proper holomorphic self-mapping $f\colon \Omega \to \Omega$ is a biholomorphism.*

The proof is based on a careful analysis of the branch locus of a proper holomorphic mapping from a pseudoconvex domain with real analytic boundary. Notice that the assumption of real analyticity is crucial here. To the best of our knowledge, it is not known whether the analog of Theorem 7.11 remains true for pseudoconvex domains with smooth boundary of finite type in the sense of D'Angelo. Certain results of this type are obtained for domains which admit some symmetries.

A domain Ω is said to be quasi-circular if there exist integers p and q, $p + q \geq 1$, such that $(e^{ip\theta}z, e^{iq\theta}w) \in \Omega$ for $\theta \in [0, 2\pi]$ whenever $(z, w) \in \Omega$. Thus, if $p = q = 1$, the domain Ω is circular; when $p = 0$ or $q = 0$, Ω is a Hartogs domain. The following result is obtained in [53, 54].

Theorem 7.12. *Let Ω be a smoothly bounded pseudoconvex quasi-circular domain of finite type in $\mathbb{C}^2$. Then every proper holomorphic self-map of Ω is a biholomorphism.*

The proof uses the scaling method (for the study of the branch locus of the map) and arguments from holomorphic dynamics.

Assumptions on regularity of the boundary can be weakened for domains with additional symmetries. For every $a \in \mathbb{C}^n$ denote by $L_a\colon \mathbb{C}^n \to \mathbb{C}^n$ the linear map $T_a z = (a_1 z_1, \dots, a_n z_n)$. Recall that a domain $\Omega \subset \mathbb{C}^n$ is called a Reinhardt (respectively, complete) domain if $T_a(\Omega) = \Omega$ for every a with $|a_j| = 1$ (respectively, $|a_j| \leq 1$), $1 \leq j \leq n$. We note that a complete description of automorphisms of a wide class of hyperbolic Reinhardt domains was obtained by Kruzhilin [106].

The following result was established by Berteloot and Pinchuk [30]:

Theorem 7.13. *Among bounded complete Reinhardt domains in $\mathbb{C}^2$, the bidiscs are the only domains that admit proper holomorphic self-mappings that are not automorphisms.*

The same work also contains a detailed description of proper holomorphic maps between complete Reinhardt domains. The general case of Reinhardt domains in $\mathbb{C}^2$ (not necessarily complete) was considered by Isaev and Kruzhilin [92]. They obtained a complete description of proper holomorphic mappings and classified all Reinhardt domains in $\mathbb{C}^2$ that admit proper holomorphic self-maps which are not biholomorphisms. A partial generalization of Theorem 7.13 to higher dimensions was obtained by Berteloot [27].

Proper holomorphic mappings between the classical Cartan domains and a wide class of Siegel domains were studied by Tumanov and Henkin [149, 150] and by Henkin and Novikov [95]. We present here one of their results.

Theorem 7.14. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be an irreducible bounded symmetric domain. Then every proper holomorphic self-map $f\colon \Omega \to \Omega$ is an automorphism of Ω.*

8. PROPER HOLOMORPHIC MAPPINGS BETWEEN REAL ANALYTIC DOMAINS

The goal of this section is to present the following results obtained by Diederich and Pinchuk in [65, 66].

Theorem 8.1. *Let $f\colon \Omega \to \Omega'$ be a proper holomorphic mapping between two bounded domains in $\mathbb{C}^n$ with real analytic boundaries. Suppose that at least one of the following conditions holds*:

(a) $n = 2$;

(b) *f extends continuously to $\overline{\Omega}$.*

Then f extends holomorphically to a neighbourhood of $\overline{\Omega}$.

When the map f is assumed to be a biholomorphism and to extend smoothly to the boundary of Ω, this result was obtained by Baouendi, Jacobowitz, and Trèves [11]. For pseudoconvex domains this was proved in any dimension and without the assumption of boundary continuity by Diederich and Fornaess [63] and by Baouendi and Rothschild [12]. In that case pseudoconvex boundaries are automatically of finite type and Condition R holds. Therefore, a proper holomorphic map f extends smoothly to $\overline{\Omega}$ by Theorem 6.5. The case $n = 2$ was also considered by Huang [97] under an additional assumption that f is continuous on $\overline{\Omega}$.

The case of condition (b) follows from a more general result.

Theorem 8.2. *Let $\Gamma \subset \Omega$ and $\Gamma' \subset \Omega'$ be real analytic closed hypersurfaces of finite type, and let $f\colon \Gamma \to \Gamma'$ be a continuous CR mapping. Then f extends holomorphically to a neighbourhood of Γ.*

The proof of these results consists of two major parts:

(1) one proves that f extends as a proper holomorphic correspondence to a neighbourhood of $\partial\Omega$;

(2) one proves that if f extends as a proper holomorphic correspondence, then it extends as a holomorphic mapping to a neighbourhood of $\partial\Omega$.

The main tool in the proof is the invariance property of Segre varieties associated with the real analytic hypersurfaces. Their behaviour near Levi degenerate points of the boundary requires a more subtle analysis. We describe now the main steps.

Let Ω be a bounded domain in $\mathbb{C}^n$ with real analytic boundary. There exists a neighbourhood W of $\partial\Omega$ and a real analytic function $\rho\colon W \to \mathbb{R}$ such that $\Omega \cap W = \{z \in W\colon \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for all $z \in \partial\Omega$ (a global defining function). Its complexification $\rho(z, \overline{w})$ is defined on a suitable neighbourhood $V \subset \mathbb{C}^{2n}$ of the diagonal $\Delta \subset W \times W$ and is holomorphic in z and antiholomorphic in w. For points $z \in \mathbb{C}^n$ we use the notation $z = ('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

Let $z^0 \in \partial\Omega$. A local holomorphic coordinate system centred at z^0 is called standard if the defining function ρ can be written in these coordinates in the form $\rho(z) = 2x_n + o(|z|)$. A pair of open neighbourhoods $U_1 \subset U_2$ (with $\overline{U_1} \subset U_2$) is called a standard pair of neighbourhoods of z^0 if it has the following properties:

(a) with respect to a suitable standard coordinate system at z^0 one has $U_2 = {}'U_2 \times U_{2n}$ with ${}'U_2$ being an open neighbourhood of $0 \in \mathbb{C}^{n-1}$ and U_{2n} an open neighbourhood on the z_n-axis;

(b) the complexification $\rho(z,\overline{w})$ is well defined on $U_2 \times U_1$ so that for each $w \in U_1$ the Segre variety $Q_w = \{z \in U_2 \colon \rho(z,\overline{w}) = 0\}$ is well defined;

(c) Q_w can be written as a graph; that is, there exists a holomorphic function $h_w('z)$ on $'U_2$ (depending antiholomorphically on w) such that

$$Q_w = \big\{('z, z_n) \in U_2 \colon\ z_2 = h_w('z)\big\}. \tag{8.1}$$

Note that every point $z^0 \in \partial\Omega$ admits a family of standard pairs of neighbourhoods such that the corresponding U_2 form a neighbourhood basis of z^0. With this notation the function $h('z,\overline{w}) := h_w('z)$ can be written as a power series $h('z,\overline{w}) = \sum_j \lambda_j(\overline{w})z^j$ with coefficients λ_j antiholomorphic on U_1. There exists an integer N (depending only on $\partial\Omega$) such that for all $z^0 \in \partial\Omega$ and any standard pair of neighbourhoods $U_1 \subset U_2$ of z^0 the coefficients $\{\lambda_j \colon |j| \le N\}$ uniquely determine Q_w. This allows us to define the structure of a finite-dimensional complex variety on the family of all Segre varieties so that the maps

$$\lambda \colon\ U_1 \ni w \mapsto Q_w \tag{8.2}$$

are finite antiholomorphic branched coverings. For any point $w \in W$, with W being a sufficiently small open neighbourhood of $\partial\Omega$, one has the following: the complex line l_w through w containing the real line passing through w and orthogonal to $\partial\Omega$ intersects the Segre variety Q_w at exactly one point ${}^{s}w$, called the symmetric point of w. For $w \in W \setminus \overline{\Omega}$ one always has ${}^{s}w \in \Omega$. The connected component of $Q_w \cap \Omega$ containing ${}^{s}w$ is denoted by ${}^{s}Q_w$ and is called the symmetric component.

The second important technical tool is provided by holomorphic correspondences. Let U and U' be open subsets of $\mathbb{C}^n$. A proper holomorphic correspondence is a closed complex analytic subset $F \subset U \times U'$ of pure dimension n such that the canonical projection $\pi \colon F \to U$ is proper. The correspondence F is called irreducible if $F \subset U \times U'$ is irreducible as an analytic set (see [47] for generalities on complex analytic sets). Let Ω and Ω' be bounded domains in $\mathbb{C}^n$ and $z^0 \in \partial\Omega$ be a boundary point. We say that f extends as a proper holomorphic correspondence to a neighbourhood U of z^0 if there exists an open set $U' \subset \mathbb{C}^n$ and an irreducible proper holomorphic correspondence $F \subset U \times U'$ such that

$$\Gamma_f \cap \{(\Omega \cap U) \times \Omega'\} \subset F,$$

where Γ_f denotes the graph of f.

One can view this as an extension of f as a multiple-valued map. Indeed, to each point $z \in U$ the correspondence F assigns a finite number of points in the target space, namely, the set $\widehat{F}(z) := \pi'(\pi^{-1}(z))$, where π' denotes the projection of F to U'.

Let $f \colon \Omega \to \Omega'$ be a proper holomorphic mapping between two bounded domains with real analytic boundaries in $\mathbb{C}^n$. Suppose that f extends as a correspondence F to a neighbourhood of the point $z^0 \in \partial\Omega$. Choose standard coordinates such that $z^0 = 0$, $f(z^0) = 0$ and standard neighbourhoods U_j and U'_j, $j = 1, 2$. Then we have the following invariance property for the Segre varieties under $\widehat{F}$.

Proposition 8.3. *For every $(w, w') \in F \cap (U_1 \times U'_1)$, the inclusion $\widehat{F}(Q_w) \subset Q'_{w'}$ holds.*

Now we can explain how to construct a holomorphic correspondence which extends the graph of f. For $\zeta \in Q_w$ we denote by ${}_{\zeta}Q_w$ the germ of Q_w at ζ. For every point $z^0 \in \partial\Omega$ in a standard coordinate system, a standard pair of neighbourhoods $U_1 \subset U_2$, and a suitably chosen open neighbourhood U' of $\partial\Omega'$, we define

$$V := \big\{(w, w') \in (U_1 \setminus \overline{\Omega}) \times (U' \setminus \overline{\Omega'}) \colon\ {}_{{}^{s}w'}Q'_{w'} \subset f(Q_w \cap \Omega)\big\}. \tag{8.3}$$

The important step is to show that V extends as an n-dimensional analytic set to a full neighbourhood of $(0, 0)$, which, in fact, is the extension of the graph of f. This is not obvious, because this requires the properness of the projection of V to $U_1 \setminus \Omega$.

The second part of the proof is given by the following

Theorem 8.4. *Let $\Omega, \Omega' \subset \mathbb{C}^n$ be bounded domains with real analytic boundaries and $f\colon \Omega \to \Omega'$ be a proper holomorphic mapping that extends as a holomorphic correspondence to a neighbourhood of $\overline{\Omega}$. Then f extends holomorphically to a (possibly smaller) neighbourhood of $\overline{\Omega}$.*

The original proof of this result [64] used the fact that f extends smoothly to pseudoconvex points of $\partial\Omega$. This result in turn used subelliptic estimates for the $\overline{\partial}$-Neumann operator. Later Pinchuk and Shafikov [128] gave a self-contained geometric proof without using the $\overline{\partial}$-methods. Further, in [67] Diederich and Pinchuk showed that for a holomorphic extension of the map f it is enough to assume that its graph extends as an analytic set of dimension n (i.e., the projection π from this set is not assumed to be proper).

9. ANALYTIC DISCS

In this section we consider a special case of proper holomorphic mappings from the unit disc to domains in $\mathbb{C}^n$. Since the unit disc does not have biholomorphic invariants, analytic discs are more flexible than holomorphic mappings between domains in $\mathbb{C}^n$ for $n > 1$. This flexibility makes them very useful in geometric complex analysis and its applications. These applications are often based on the existence of analytic discs with boundaries in prescribed CR manifolds. We discuss here some important results of this type.

9.1. Gromov's theorem. Consider the standard symplectic form on $\mathbb{C}^n$:

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

A real submanifold E of dimension n in $\mathbb{C}^n$ is called *Lagrangian* if $\omega|E = 0$. It is easy to see that every Lagrangian manifold is totally real, but the class of totally real manifolds is larger. The following result is due to Gromov [89].

Theorem 9.1. *Let E be a smooth compact Lagrangian submanifold in $\mathbb{C}^n$. Then there exists a nonconstant analytic disc smooth on $\overline{\mathbb{D}}$ with the boundary attached to E.*

This theorem has deep applications in symplectic geometry (see, for example, [6]). Note that one can view it as a (partial) generalization of the Riemann mapping theorem. Indeed, when $n = 1$, every real curve is Lagrangian.

From the analytic point of view the problem of constructing an analytic disc with the boundary glued to E can be regarded as a Riemann–Hilbert type boundary value problem with nonlinear boundary data (given by E). We sketch the main steps of Gromov's approach following the work of Alexander [4], who gave a simplified version of Gromov's approach in the case of $\mathbb{C}^n$. We note that the original methods of Gromov lead to considerably more general results.

Step 1: *Manifolds of discs and elliptic estimates.* Fix a point $p \in E$ and fix also a noninteger $r > 1$. Consider the set of pairs

$$\mathcal{F} = \bigl\{f \in C^{r+1}(\mathbb{D}, \mathbb{C}^n)\colon\ f(\partial\mathbb{D}) \subset E,\ f(1) = p\bigr\}. \tag{9.1}$$

Denote by F an open subset of $\mathcal{F}$ which consists of those f that are homotopic to a constant map $f^0 \equiv p$ in $\mathcal{F}$. It is well known that F is a complex Banach manifold. Denote by G the complex Banach space of all C^r maps $g\colon \mathbb{D} \to \mathbb{C}^n$. Set $H = \{(f, g) \in F \times G\colon \partial f/\partial\overline{\zeta} = g\}$. Then H is a connected submanifold of $F \times G$.

For $0 < t < 1$, let $\mathbb{D}_t := t\mathbb{D}$ and $\mathbb{D}_t^+ := t\mathbb{D} \cap \{\operatorname{Im}\zeta > 0\}$.

Lemma 9.2. *Let* $f_k\colon (\mathbb{D}_t^+, \partial\mathbb{D}_t^+ \cap \mathbb{R}) \to (\mathbb{C}^n, E)$ *be maps of class* C^{r+1} *that converge uniformly to* $f\colon (\mathbb{D}_t^+, \partial\mathbb{D}_t^+ \cap \mathbb{R}) \to (\mathbb{C}^n, E)$. *Suppose that the sequence* $g_k = \partial f_k/\partial\overline{\zeta}$ *converges in* $C^r(\mathbb{D}_t^+)$ *to* $g \in C^r(\mathbb{D}_t^+)$. *Then for every* $\tau < t$ *one has* $f \in C^{r+1}(\mathbb{D}_t^+)$ *and* $\{f_k\}$ *converges to* f *in* $\mathbb{D}_\tau^+$ *in the* C^{r+1}*-norm.*

Denote by $T_{\mathbb{D}}f = (2\pi i)^{-1} f * (1/\zeta)$ the Cauchy–Green integral on $\mathbb{D}$. Recall the classical regularity property of the Cauchy–Green integral: for every noninteger $s > 0$ the linear map $T_{\mathbb{D}}\colon C^s(\mathbb{D}) \to C^{s+1}(\mathbb{D})$ is bounded. The proof of Lemma 9.2 given in [4] is based on the standard elliptic "bootstrapping" argument employing the above regularity of the Cauchy–Green operator and elementary estimates for the harmonic measures. Notice that this proof is purely local, i.e., all estimates and the convergence are established in a neighbourhood of a given boundary point of a disc. The global statement is the following

Lemma 9.3. *Suppose that a sequence* $\{f_k\}$ *in* $\mathcal{F}$ *converges to a continuous mapping* $f\colon (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^n, E)$ *uniformly on* $\overline{\mathbb{D}}$ *and* $g_k := \partial f_k/\partial\overline{\zeta}$ *converge in* $C^r(\mathbb{D})$ *to* $g \in C^r(\mathbb{D})$. *Then* $f \in C^{r+1}(\mathbb{D})$ *and* $\{f_k\}$ *converges to* f *in* $\mathcal{F}$ *after possibly passing to a subsequence.*

Considering a finite covering of $\partial\mathbb{D}$ by such neighbourhoods, we obtain C^{r+1} convergence in a neighbourhood of $\partial\mathbb{D}$. The convergence in the interior of $\mathbb{D}$ follows, since $f_k = T_{\mathbb{D}}g_k + h_k$ and the bounded sequence $\{h_k\}$ of holomorphic functions is a normal family.

Notice that the above boundary regularity and convergence results for analytic discs are quite similar to the tools used in the proof of Fefferman's mapping theorem.

Step 2: *Renormalization and scaling.* The canonical projection $\pi\colon H \to G$ given by $\pi(f,g) = g$ is a map of class C^1 between two Banach manifolds. It is known [4, 89] that π is a Fredholm map of index 0 and the constant map f^0 is a regular point for π.

The crucial property of π is proved in [4]: *the map* π *is not surjective.* Now, arguing by contradiction, suppose that there exists no nonconstant analytic disc of class $C^{r+1}(\mathbb{D})$ attached to E; then $\pi^{-1}(0) = \{f^0\}$. It follows that $0 \in G$ is a regular value of π. If π is proper, then Gromov's argument based on the Sard–Smale theorem implies surjectivity of π (see [4]), a contradiction. Thus, it remains to show that $\pi\colon H \to G$ is proper.

Suppose by contradiction that π is not proper. Then there exists a sequence $\{(f_k, g_k)\} \subset H$ such that $g_k \to g$ in G but f_k diverge in F. For every k consider the function q_k defined by $q_k(\zeta) = T_{\mathbb{D}}g_k(\zeta)$ for $\zeta \in \overline{\mathbb{D}}$ and $q_k(\zeta) = 0$ for $\zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Then $q_k \to q = T_{\mathbb{D}}g$ in $C^{r+1}(\overline{\mathbb{D}}, \mathbb{C}^n)$ and $f_k = q_k + h_k$, where $h_k \in C^{r+1}(\mathbb{D}, \mathbb{C}^n)$ and h_k is holomorphic on $\mathbb{D}$. We have $f_k(\partial\mathbb{D}) \subset E$ and q_k are uniformly bounded since g_k are; we conclude that $h_k|_{\partial\mathbb{D}}$ are uniformly bounded. By the maximum principle the functions h_k are uniformly bounded on $\overline{\mathbb{D}}$. Hence, f_k are uniformly bounded.

Set $M_k = \sup_{\mathbb{D}}|h_k'(\lambda)|$. Since $h_k \in C^r(\mathbb{D}, \mathbb{C}^n)$ and $r > 1$, the constants M_k are finite for every k. If $\{M_k\}$ contains a bounded subsequence, then a subsequence of $\{h_k\}$ converges uniformly on $\mathbb{D}$. Then a subsequence of $\{f_k\}$ converges uniformly, and by Lemma 9.3 it converges in $\mathcal{F}$, a contradiction. Thus, we may suppose that $M_k \to \infty$. The key idea of [4] is to apply a renormalization argument which is essentially a version of the scaling argument.

There exists $\lambda_k \in \partial\mathbb{D}$ with $M_k = |h'(\lambda_k)|$. Taking a subsequence if necessary, suppose that $\lambda_k \to \lambda^*$. Set $z_k = (1 - M_k^{-1})\lambda_k \in \mathbb{D}$ and consider the renormalization sequence of the Möbius maps

$$\phi_k(\lambda) = \frac{\lambda + z_k}{1 + \overline{z}_k\lambda}.$$

Set $\widetilde{f}_k = f_k \circ \phi_k$, $\widetilde{q}_k = q_k \circ \phi_k$, and $\widetilde{h}_k = h_k \circ \phi_k$. It is proved in [4] that after extracting a subsequence, the sequences $\{\widetilde{q}_k\}$ and $\{\widetilde{h}_k\}$ converge uniformly on compact sets in $\overline{\mathbb{D}} \setminus \{-\lambda^*\}$ to a constant map c and a holomorphic map $\widetilde{h}$, respectively.

Notice that since q_k converge in $C^{r+1}(\overline{\mathbb{D}})$, the sequence $\{\widetilde{q}_k\}$ converges on compact sets in $\overline{\mathbb{D}} \setminus \{-\lambda^*\}$ in this norm. Since Lemma 9.3 is local, it applies and gives the convergence of $\{\widetilde{f}_k\}$ to $\widetilde{f}$ in the C^{r+1}-norm on compact sets in $\overline{\mathbb{D}} \setminus \{-\lambda^*\}$ as well. Then again the argument of [4] shows that $|\widetilde{h}'_k(\lambda_k)|$ converges to $1/2 = |\widetilde{h}'(\lambda^*)|$. Hence, $\widetilde{f}$ is a nonconstant disc of class C^{r+1}. By the boundary regularity theorem for analytic discs, we conclude that f is of class C^∞ on $\overline{\mathbb{D}} \setminus \{1\}$.

Furthermore, since E is a Lagrangian manifold, it is easy to see that the disc $\widetilde{f}$ has bounded area. This is due to the fact that for an analytic disc the area (induced by the Euclidean structure) coincides with the symplectic area $[\mathbb{D}](f^*\omega)$ (where $[\mathbb{D}]$ is the current of integration over $\mathbb{D}$). Then [4, Theorem 2] implies that $f\colon \mathbb{D} \setminus f^{-1}(E) \to \mathbb{C}^n \setminus E$ is a proper map. But then $\widetilde{f}$ extends smoothly to a neighbourhood of the point 1 (see Proposition 2.6) and so is smooth on $\overline{\mathbb{D}}$. This contradicts our assumption of nonexistence of nonconstant analytic discs attached to E, and the theorem is proved.

The assumption that E is Lagrangian is crucial in Gromov's theorem. Alexander [5] constructed a totally real torus T^2 in $\mathbb{C}^2$ which does not contain the boundary of any analytic disc. However, in this example one can attach to T^2 the boundary of some Riemann surface (an annulus). This phenomenon was recently studied by Duval and Gayet [71] for certain classes of totally real tori in $\mathbb{C}^2$. Their approach uses the results of Bedford and Gaveau [17], Bedford and Klingenberg [18], and Kruzhilin [107] on filling topological 2-spheres, contained in compact strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$, with Levi flat hypersurfaces. The filling is provided by a one-parameter family of analytic discs attached to the sphere. These results have many other applications, including those in symplectic topology.

9.2. Discs in pseudoconvex domains. Another approach to the extension of the Riemann mapping theorem concerns construction of proper holomorphic discs in domains in $\mathbb{C}^n$. We begin with the following result of Forstnerič and Globevnik [84].

Theorem 9.4. *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Then for every point $p \in \Omega$ there exists an analytic disc $f\colon \mathbb{D} \to \Omega$ smooth on $\overline{\mathbb{D}}$ such that $p = f(0)$ and $f(\partial\mathbb{D}) \subset \partial\Omega$.*

In fact, even stronger results in this direction have been obtained (we refer the reader to [70] for a detailed account). The proof of Theorem 9.4 can be described as follows. Consider a global defining function ρ of Ω. The idea is to construct an analytic disc attached to a suitable noncritical sublevel set of ρ. When such a sublevel is a small deformation of a ball around p, this can be achieved by the implicit function theorem. The main part of the proof consists of two major steps. First, an approximate solution of the Riemann–Hilbert type boundary value problem allows one to construct a homotopy on the space of analytic discs attached to the noncritical level sets of ρ. The second step is a careful analysis of the Morse geometry of a critical level set of ρ, which allows one to push an analytic disc on the post-critical level set. Combining these two tools, we can begin with a small disc attached to some noncritical level and then deform it through other levels to a global disc attached to the boundary.

For strictly convex domains even stronger results can be obtained. This theory was developed by Lempert [108]. Let Ω be a strictly convex domain in $\mathbb{C}^n$ (this means the real Hessian of the boundary is positive definite; in particular, Ω is strictly pseudoconvex). Fix a point $p \in \Omega$. Then for every tangent vector V at p there exists a unique analytic disc f centred at p in the direction of V which is extremal for the Kobayashi–Royden metric of Ω. The condition of extremality means that the infimum in the definition of the metric is achieved on this disc. It turns out that f is smooth up to the boundary and its boundary is attached to $\partial\Omega$. Moreover, f admits a holomorphic lift which is attached to the projectivization of the holomorphic tangent bundle of $\partial\Omega$. Since $\mathbb{P}H(\partial\Omega)$ is a totally real manifold, f satisfies a Riemann–Hilbert type boundary value problem. When Ω is a small deformation of the unit ball, this problem can be easily solved by the implicit function theorem. The general case requires more advanced tools provided by the continuity method. It consists of two major steps: the implicit function theorem for the linearized Riemann–Hilbert boundary value problem and a priori estimates (here the assumption of strict convexity is used).

In the case of the unit ball Lempert's discs through the origin are just linear and are given by the intersection of complex lines with the ball. It turned out that in the general case the geometry of extremal discs through any point p is similar: they form a singular foliation of Ω with a unique singularity at p. This allows one to construct the "Riemann mapping" from Ω to $\mathbb{B}^n$ which is holomorphic along every extremal disc through a fixed point p and preserves the contact structure of the boundary.

Lempert's theory has many applications. For example, it provides an independent proof of Fefferman's mapping theorem. Furthermore, extremal discs form a very useful family of biholomorphic invariants, which leads to a solution of the biholomorphic equivalence problem [109]. The logarithm of the Euclidean norm of the above "Riemann mapping" gives a solution of the complex Monge–Ampère equation with a logarithmic pole at p; it can also be viewed as a higher dimensional analog of the Green function. Donaldson used a similar approach [69] in order to construct regular solutions of the Dirichlet problem for a certain class of complex Monge–Ampère equations.

10. POSITIVE CODIMENSION

In this section we consider the properties of holomorphic mappings $f\colon \Omega \to \Omega'$, where $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^N$ with $1 < n < N$ (the case of *positive codimension*). These maps do not have flexibility of analytic discs, since the boundary of the source domain has intrinsic geometry. Nevertheless, the case of positive codimension is considerably more flexible than the equidimensional one. This is illustrated by the following result due to Forstnerič [76] and Løw [111].

Theorem 10.1. *Let Ω be a bounded strictly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. There is an integer N_1 such that for every $N \ge N_1$ there exists a proper holomorphic mapping $f\colon \Omega \to \mathbb{B}^N$. Some of these embeddings extend continuously to $\overline{\Omega}$, but there also exist embeddings that are not continuous on $\partial\Omega$.*

In particular, a direct analog of Fefferman's mapping theorem is not true in the case of positive codimension.

Note that the tools of the Moser theory or Cartan–Chern theory do not seem to be appropriate in this case. This is one of the reasons why the study of the rigidity phenomenon in positive codimension is a difficult problem. One of the main tools here is the geometric reflection principle based on the geometry of Segre varieties, which seems to admit some generalization to the case of positive codimension.

Forstnerič [82] showed that most generic real analytic CR manifolds of positive CR dimension are not locally holomorphically embeddable to the germ of any generic real algebraic CR manifold of the same real codimension. One of the principal facts is that an analog of the Poincaré–Alexander phenomenon holds for CR mappings between real spheres of positive codimension if the initial regularity of a CR mapping is sufficiently high. Forstnerič [77] proved that such a CR mapping extends to a rational mapping with un upper bound on the degree (depending on the codimension). Similar results are obtained for holomorphic mappings between real algebraic CR manifolds (see, for example, [8, 52, 112, 161]). However, the extension of the Poincaré–Alexander phenomenon to the real analytic category meets difficulties. The following result is due to Pinchuk [123].

Theorem 10.2. *Let Γ be a (connected) real analytic strictly pseudoconvex hypersurface in $\mathbb{C}^n$, $n > 1$. Assume that f is a smooth CR mapping in a neighbourhood U of a point $p \in \Gamma$ such that $f(U) \subset \partial\mathbb{B}^N$ with $n \le N$. Then f extends as a holomorphic mapping along any path in Γ.*

The proof is based on the analytic reflection principle. Currently there is no extension of this result to the case when the sphere $\partial\mathbb{B}^n$ is replaced with a real analytic strictly pseudoconvex hypersurface. Furthermore, even in the case of the local Schwarz-type reflection principle many basic questions remain open.

Finally, Forstnerič [78] established the following

Theorem 10.3. *Let $f\colon \Gamma \to \Gamma'$ be a smooth CR mapping between real analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively, $n \le N$. Then there exists an open dense subset $O \subset \Gamma$ such that f extends holomorphically to a neighbourhood of every point of O.*

A natural question is whether the above set O coincides with the whole Γ. The following result was obtained in [129].

Theorem 10.4. *Let $f\colon \Gamma \to \Gamma'$ be a smooth CR mapping between real analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively, and $n \le N \le 2n$. Then f extends holomorphically to a neighbourhood of every point in Γ.*

The proofs of Theorems 10.3 and 10.4 are based on the geometric reflection principle and the study of Segre varieties. To the best of our knowledge, it is not known whether the condition $N \le 2n$ in Theorem 10.4 can be dropped.

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