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# ANALYTIC DIFFERENTIAL EQUATIONS AND SPHERICAL REAL HYPERSURFACES

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**Abstract**

We establish an injective correspondence $M \longrightarrow \mathcal{E}(M)$ between real-analytic nonminimal hypersurfaces $M \subset \mathbb{C}^2$, spherical at a generic point, and a class of second order complex ODEs with a meromorphic singularity. We apply this result to the proof of the bound $\dim \mathfrak{hol}(M,p) \leq 5$ for the infinitesimal automorphism algebra of an *arbitrary* germ $(M,p) \not\sim (S^3,p')$ of a real-analytic Levi nonflat hypersurface $M \subset \mathbb{C}^2$ (the Dimension Conjecture). This bound gives the proof of the dimension gap $\dim \mathfrak{hol}(M,p) = \{8,5,4,3,2,1,0\}$ for the dimension of the automorphism algebra of a real-analytic Levi nonflat hypersurface. As another application we obtain a new regularity condition for CR-mappings of nonminimal hypersurfaces, that we call *Fuchsian type,* and prove its optimality for the extension of CR-mappings to nonminimal points.

We also obtain an existence theorem for solutions of a class of singular complex ODEs.

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## 1. Introduction

The goal of this paper is to give solution to a number of previously open problems in CR-geometry, including an old question of H. Poincaré, by introducing a new technique when a CR-manifold under consideration is replaced by an appropriate holomorphic dynamical system. By doing so we reduce the original problem to a classical setting in local holomorphic dynamics. Using this approach the authors [**33**] proved recently that for any positive CR-dimension and CR-codimension the holomorphic moduli space in CR-geometry is bigger than the formal one. We describe below the CR-geometry problems addressed in the paper, and briefly explain our dynamical approach. To outline the parallels between CR-geometry and complex dynamical systems we summarize the connection between the geometric objects and the corresponding dynamical analogues in a table at the end of this introduction.

Let $M, M' \ni 0$ be two real-analytic hypersurfaces in the complex space $\mathbb{C}^2$. A local biholomorphic mapping $\mathcal{F} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ is called *a holomorphic equivalence between* $(M, 0)$ *and* $(M', 0)$*,* if $F(M) \subset M'$. In 1907 H. Poincaré formulated his *problème local* [**43**]: given two germs of real-analytic hypersurfaces $M, M' \subset \mathbb{C}^2$, find all local holomorphic equivalences between them. The discovery of Poincaré was that the problem is highly nontrivial due to the fact that germs of Levi nondegenerate hypersurfaces in $\mathbb{C}^2$ possess biholomorphic invariants. That makes two germs in general position holomorphically inequivalent. Another discovery of Poincaré was that the local automorphism group $\mathrm{Aut}\,(M,0)$ of a Levi nondegenerate hypersurface is finite dimensional and is always a subgroup in the stability group $\mathrm{Aut}\,(S^3,o)$ of a point $o$ lying in the 3-dimensional sphere $S^3 \subset \mathbb{C}^2$. For the pseudogroup of local self-mappings of a Levi nondegenerate hypersurface (or, alternatively, for the well-defined associated infinitesimal automorphism algebra $\mathfrak{hol}\,(M,0)$) Poincaré gives the bound $\dim \mathfrak{hol}\,(M,0) \leq \dim \mathfrak{hol}\,(S^3,o) = 8$. The considerations of Poincaré were based on the existence of a "model" Levi nondegenerate hypersurface, namely, the quadric $\mathcal{Q} = \{\mathrm{Im}\, w = |z|^2\} \cong S^3$. Ideas of Poincaré were developed and generalized in the work of E. Cartan [**10**], N. Tanaka [**49**], S. Chern and J. Moser [**12**], who obtained a complete solution for the local holomorphic equivalence problem for real-analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^n$, $n \geq 2$.

Today, after more than a century, *problème local* is still very far from being solved completely. We outline below some recent results and explain the difficulties in completing the problem.

For hypersurfaces in $\mathbb{C}^2$ with Levi degeneracies satisfying the finite type condition (see, e.g., [**4**]), the equivalence problem was studied by V. Beloshapka, V. Ezhov and M. Kolar and completed in the work of Kolar [**29**]. The problem in the finite type case was treated in the spirit of Poincaré by using *models*, i.e., hypersurfaces defined by $\mathrm{Im}\, w = P_k(z,\bar z)$, where $P_k(z,\bar z)$ is a nonzero homogeneous polynomial of degree $k \geq 3$ without harmonic terms. These models allow one to obtain a formal normal form for finite type real-analytic hypersurfaces $M \subset \mathbb{C}^2$. Even though such a normal form can be divergent (see [**30**]), convergence results for formal CR-equivalences (see, e.g., [**5**]) show that such a normal form is a biholomorphic invariant and thus a solution for the holomorphic equivalence problem. By relaxing the finite type condition one comes to the consideration of a significantly more difficult to analyze class of the so-called *nonminimal* hypersurfaces (the term is coined in [**51**]), that is real hypersurfaces $M$ containing a complex hypersurface $X$. The main obstruction for solving *problème local* in the nonminimal case is perhaps hidden in the fact that polynomial hypersurfaces arising from the defining equation of a nonminimal hypersurface can *no longer* be considered as models in the sense of Poincaré-Chern-Moser. For example, in the class of nonminimal hypersurfaces $\bigl\{\mathrm{Im}\, w = (\mathrm{Re}\, w)\psi(|z|^2),\, \psi(0)=0,\, \psi'(0)\neq 0\bigr\}$, all of which contain the complex hypersurface $X = \{w=0\}$, any polynomial model has the isotropy group of dimension 2, while the hypersurface $\mathrm{Im}\, w = (\mathrm{Re}\, w)\tan\left(\frac{1}{2}\arcsin|z|^2\right)$ has the isotropy group of dimension 5 (see [**8**], [**31**]). A recent result of the authors [**33**] showing that formal equivalences between nonminimal hypersurfaces can be actually divergent, proves, in particular, that a formal normal form can no longer be a solution for the equivalence problem for nonminimal hypersurfaces, which further illustrates the difficulties for this class of hypersurfaces. In fact, even the class of nonminimal hypersurfaces *spherical at a generic point* appears to be highly nontrivial (we refer here to the work [**34**, **17**, **6**, **31**, **32**, **33**] of V. Beloshapka, P. Ebenfelt, M. Kolar, Kowalski, B. Lamel, D. Zaitsev and the authors), as it is not even known whether the moduli space for this class of hypersurfaces is finite dimensional.

One of the goals of the present paper is to give a complete solution for the automorphism version of Poincaré's *problème local.* We first give a solution in the nonminimal case, more precisely, we prove the following

**Theorem 1.** *Let $M \subset \mathbb{C}^2$ be a real-analytic nonminimal at the origin Levi nonflat hypersurface. Then the dimension of its infinitesimal automorphism algebra satisfies the bound*

$$dim\, \mathfrak{hol}\,(M,0) \leq 5. \tag{1.1}$$

The previous example of the hypersurface $\operatorname{Im} w = (\operatorname{Re} w) \tan\left(\frac{1}{2} \arcsin |z|^2\right)$ shows that the bound in Theorem 1 is in fact sharp. As a corollary, we obtain the following "dimension gap" phenomenon, solving the *problème local* (in the automorphism interpretation) completely.

**Corollary 1** (see Theorem 3.11). *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, $0 \in M$, and let $M$ be Levi nonflat. Then $\mathfrak{hol}\,(M,0)$ is isomorphic to a subalgebra in $\mathfrak{hol}\,(S^3,o) \simeq \mathfrak{su}(2,1)$. Moreover, the bound $dim\, \mathfrak{hol}\,(M,0) \leq 5$ holds unless $(M,0)$ is biholomorphic to $(S^3,o)$ for $o \in S^3$. In particular, the dimension gap $dim\, \mathfrak{hol}(M,0) \in \{8,5,4,3,2,1,0\}$ holds for all possible dimensions of the infinitesimal automorphism algebra of real-analytic Levi nonflat hypersurfaces $M \subset \mathbb{C}^2$.*

Corollary 1 should be compared with various dimension gap phenomena in differential geometry, in particular, for isometries of Riemannian manifolds (see, e.g., S. Kobayashi [**28**]), or for automorphism groups of Kobayashi hyperbolic manifolds (see, e.g., A. Isaev [**23**, **24**] and references therein). An interesting parallel here is given by the fact that the maximal dimension 8 for the automorphism group of a two-dimensional hyperbolic manifold is realized only for the special case of the 2-ball $\mathbb{B}^2 \subset \mathbb{C}^2$, while for the automorphism algebra of a real-analytic Levi nonflat hypersurface $M \subset \mathbb{C}^2$ the maximal dimension 8 is realized only for the 3-sphere $S^3 = \partial\mathbb{B}^2$.

We can further formulate

**Corollary 2.** *Let $M \subset \mathbb{C}^2$ be a real-analytic Levi nonflat hypersurface, $M \ni 0$. Suppose that the stability group $Aut(M, 0)$ is a Lie group in the natural topology. Then $dim\, Aut(M, 0) \leq 5$.*

The example of the 3-sphere $S^3 \subset \mathbb{C}^2$ (or the previous example of the nonminimal hypersurface $\operatorname{Im} w = (\operatorname{Re} w) \tan\left(\frac{1}{2} \arcsin |z|^2\right)$) show that the bound in Corollary 2 is sharp. For the most recent results on Lie group structures for automorphism groups of real-analytic CR-manifolds we refer to the work [**26**, **27**] of R. Juhlin and B. Lamel.

The assertions of Theorem 1 and Corollaries 1 and 2 are known as different versions of the Dimension Conjecture, see the survey [**7**] and also [**17**], [**8**], and [**31**] for partial results in this direction. For various corollaries of Theorem 1 concerning infinitesimal automorphism algebras of real-analytic germs, as well as intermediate results, we refer the reader to Section 3. In particular, Theorem 3.7 gives a curious description of the infinitesimal automorphism algebra of a nonminimal spherical hypersurface as a subalgebra in the centralizer of a special element $\sigma \in \operatorname{Aut}(\mathbb{CP}^2)$.

Another question addressed in the paper is the analytic continuation problem for a germ of a biholomorphism between real-analytic hypersurfaces $M, M' \subset \mathbb{C}^n$. The question goes back to another remarkable result of Poincaré in [**43**], which states that a local holomorphic equivalence $\mathcal{F} : (S^3, o) \longrightarrow (S^3, o')$ extends to a global linear-fractional automorphism of the 2-ball $\mathbb{B}^2 \subset \mathbb{C}^2$. The result of Poincaré was generalized by S. Pinchuk [**41**], who proved that if a real-analytic hypersurface $M \subset \mathbb{C}^n$ is strictly pseudoconvex, then a local holomorphic equivalence $\mathcal{F} : (M, p) \longrightarrow (S^{2n-1}, o)$ extends locally biholomorphically along any path $\gamma \subset M$, $\gamma \ni p$ (for $M = S^{2n-1} \subset \mathbb{C}^n$ the result was also obtained by H. Alexander [**1**]). Another generalization of Poincaré's Theorem was obtained by Pinchuk in [**42**] by considering instead of $S^{2n-1}$ a compact, *nonspherical*, strictly pseudoconvex real analytic hypersurface in the target space. Further generalizations in this direction were obtained by the school of A. Vitushkin, using the convergence of the Chern-Moser normal form (see [**53**] and references therein) and also in [**45**, **44**, **21**] by using extension along Segre varieties. Note that in all cited papers the hypersurface $M$ in the preimage was assumed to be minimal. However, as shown in the earlier paper [**32**] of the authors, when $M$ is nonminimal and $M'$ is the simplest possible (namely, $M'$ is a hyperquadric in $\mathbb{CP}^n$) the possibility to extend the germ of a biholomorphic mapping $\mathcal{F} : (M, p) \longrightarrow (M', p')$ analytically along a path $\gamma \subset M$, $\gamma \ni p$ *fails* to hold in general, if the path $\gamma$ intersects the complex hypersurface $X$, contained in $M$. The difficulty here is that neither the Chern-Moser-type technique (in view of the absence of a convergent normal form), nor the technique of extension along Segre varieties (in view of the fact that $Q_p \cap X \neq \emptyset$ implies $p \in X$) can be used to extend a mapping to nonminimal points in $M$. However, it was shown in [**32**] that *if* $M \setminus X$ *is Levi nondegenerate and* $X \ni 0$, *then one can choose an open set* $U \subset \mathbb{C}^n$, $U \ni 0$ *in such a way that the desired analytic extension holds (as a mapping into* $\mathbb{CP}^n$*) for any choice of a point* $p \in (U \setminus X) \cap M$ *and a path* $\gamma \subset U \setminus X$, $\gamma \ni p$ (note that $\gamma$ here need not to lie in $M$). Since $U \setminus X$ is not simply-connected, such an extension can branch about the complex locus $X$, which forms the first type of obstructions for extending a mapping into a quadric to the complex locus $X$ (see various examples provided in [**32**]). We say that the resulting (multiple-valued) analytic mapping $\mathcal{F} : U \setminus X \longrightarrow \mathbb{CP}^n$ *is associated with* $M$ (this object is defined uniquely up to a composition with an element $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$). Surprisingly, the authors found an example (see Example 6.7 in Section 6) where a local biholomorphic mapping $\mathcal{F}_0 : (M, p) \longrightarrow (S^3, o)$ of a nonminimal hypersurface $M \subset \mathbb{C}^2$ at a Levi nondegenerate point $p$ does *not* extend holomorphically to the complex locus $X$, even though the associated mapping $\mathcal{F}$ does not branch about $X$. The latter example made the extension/no extension dichotomy particularly intriguing, and also showed the existence of another type of obstruction for analytic extension to nonminimal points.

Our second main result is the discovery of the *non-Fuchsian type* condition for a hypersurface $M \subset \mathbb{C}^2$ (see Definition 1.1 below) as the second type of obstruction and the proof of the fact that no further obstructions exist beside the two mentioned previously. We formulate the results in detail below.

Let $M \subset \mathbb{C}^2$ be a real-analytic nonminimal at the origin Levi nonflat hypersurface, and $U \ni 0$ be a polydisc. We say that $M$ *is given in* $U$ *in prenormal coordinates* if the defining equation of $M \cap U$ is of the form

$$v = u^m \left( \pm |z|^2 + \sum_{k,l \geq 2} \Phi_{kl}(u) z^k \bar{z}^l \right), \tag{1.2}$$

where $z = x + iy$, $w = u + iv$ denote the coordinates in $\mathbb{C}^2$ and $\Phi_{kl}(u)$ are analytic near the origin functions. The complex locus for $M$ in this case is given by $X = \{w = 0\}$. Depending on the sign in (1.2) we call $M$ *positive* or *negative* respectively. Examples in Section 2 below show that prenormal coordinates for a nonminimal hypersurface fail to exist in general. However, Theorem 3.1 (see Section 3) shows that *prenormal coordinates always exist for every real-analytic nonminimal at the origin and spherical outside the complex locus hypersurface.*

For a nonminimal hypersurface, given in prenormal coordinates, we first prove the following geometric criterion for the analytic continuation of a mapping into a sphere.

**Theorem 2.** *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, containing a complex hypersurface $X \ni 0$, which is Levi nondegenerate and spherical in $M \setminus X$. Suppose that $M$ is given in some polydisc $U = \{|z| < \delta\} \times \{|w| < \epsilon\}$ in prenormal coordinates. Then a local biholomorphic mapping $\mathcal{F} : (M,p) \longrightarrow (S^3,p')$, $p \in (M\setminus X)\cap U$, $p' \in S^3$, extends to $X$ holomorphically if and only if for each Segre variety $Q_s$, $s \in U$, which is not a "horizontal" line $\{w = const\}$, there exists a holomorphic graph*

$$\tilde{Q}_s = \left\{(z,w) \in \mathbb{CP}^1 \times \{|w| < \epsilon\} : z = h_s(w)\right\},\ h_s \in \mathcal{O}\left(\{|w| < \epsilon\}\right)$$

*(called the extension of $Q_s$), such that $Q_s = \tilde{Q}_s \cap U$.*

We next formulate the crucial

**Definition 1.1.** Suppose that $M$ satisfies the conditions of Theorem 2. We say that $M$ *is of Fuchsian type at the origin,* if its defining function (1.2) satisfies

$$\text{(1.3)} \qquad \mathrm{ord}_0\Phi_{22} \geq m-1,\ \mathrm{ord}_0\Phi_{33} \geq 2m-2,\ \mathrm{ord}_0\Phi_{23} \geq \frac{3}{2}(m-1),$$

where $\mathrm{ord}_0$ denotes the order of vanishing of a function at the origin. If the conditions (1.3) fail to hold, we say that $M$ *is of non-Fuchsian type.*

We emphasize that the Fuchsian type condition holds automatically if $m = 1$, and fails to hold in general for $m > 1$. It is shown in Section 6 that the property of being Fuchsian is independent of the choice of prenormal coordinate system.

**Theorem 3.** *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, containing a complex hypersurface $X \ni 0$, which is Levi nondegenerate and spherical in $M \setminus X$, $U$ a sufficiently small neighbourhood of the origin, $p \in (M \setminus X) \cap U$, and let $\gamma$ be a generator of $\pi_1(U \setminus X)$, $p \in \gamma$. Suppose that $M$ is of Fuchsian type. Then a local biholomorphic mapping $\mathcal{F}_0 : (M,p) \longrightarrow (S^3,p')$, $p' \in S^3$, extends to $X$ holomorphically if and only if its analytic extension $\mathcal{F} : U \setminus X \longrightarrow \mathbb{CP}^2$ does not branch along $\gamma$.*

It is shown in Section 6 that the Fuchsian type condition in Theorem 3 is in a sense optimal. We also note that Theorem 3 demonstrates the difference between the geometry of 1-nonminimal and $m$-nonminimal hypersurfaces with $m > 1$ respectively. This difference became apparent already in the work of P. Ebenfelt [**16**], where the analyticity of CR-mappings from 1-nonminimal hypersurfaces was proved. It also appeared in the paper [**33**] of the authors, where it was shown that formal CR-mappings between $m$-nonminimal hypersurfaces with $m > 1$ can be divergent (while for $m = 1$ formal CR-mappings are always convergent, as shown by R. Juhlin and B. Lamel in [**27**]). At the end of Section 3 we formulate a conjecture on universality of the Fuchsian type condition as a regularity condition for mappings from nonminimal hypersurfaces.

As an intermediate step in the proof of Theorem 3, we prove the following existence theorem for singular ODEs: *a singular holomorphic ODE*

$$z''=\frac{1}{w}P(z,w)z'+\frac{1}{w^2}Q(z,w),\quad P,Q\in\mathcal{O}(\{|z|<\delta\}\times\{|w|<\epsilon\}),\tag{1.4}$$

*such that* $Q(z_0,0)=0$*, for some* $|z_0|<\delta$*, has a holomorphic in a neighbourhood of the origin solution* $z=h(w)$ *with* $h(0)=z_0$*, provided that no local solution of it admits a multiple-valued extension to an annulus* $\{\epsilon'<|w|<\epsilon''\}$ *with* $0<\epsilon'<\epsilon''<\epsilon$ (see Theorem 3.5 below).

The *nonlinear* complex ODE (1.4) after the substitution $u:=z'w$ can be rewritten as the first order system

$$\begin{cases} wz'=u,\\ wu'=(1+P(z,w))u+Q(z,w),\end{cases}\tag{1.5}$$

for which the right-hand side vanishes for $z=z_0, u=0, w=0$. This is a particular case of the *Briot-Bouquet type ODEs.* These are first order singular holomorphic ODE systems of the form $wz'=A(z,w)$, $z\in\mathbb{C}^n, w\in\mathbb{C}, A(0)=0$ with $A:\mathbb{C}^n\times\mathbb{C}\longrightarrow\mathbb{C}^n$ holomorphic near the origin. Briot-Bouquet type ODEs can be described as nonlinear generalizations of Fuchsian ODEs. They are known to have a holomorphic solution under the additional assumption that the linearization matrix $\frac{\partial A}{\partial z}(0)$ has no eigenvalues $k\in\mathbb{Z}, k>0$ (nonresonant case, see [**35**]). In the resonant case a holomorphic solution fails to exist in general (a simple example is given by the scalar equation $wz'=z+w$). It is easy to check that the "no-monodromy" assumption in Theorem 3.5 does not imply the "no-resonance" condition, and vice versa, so the assertion of Theorem 3.5 is nontrivial. To the best of our knowledge, the result is new (see, e.g., the recent surveys [**35**], [**20**] and references therein).

The main tool of the paper is a development, in the *Levi degenerate* case, of the fundamental connection between CR-geometry and the geometry of completely integrable systems of complex PDEs, first observed by E. Cartan [**10**] and B. Segre [**46**]. In particular, the geometry of real-analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^2$ is closely related to that of (nonsingular!) second order complex ODEs, as discussed in Section 2. For modern treatment of the connection in the nondegenerate case we refer to earlier work [**47**, **48**, **18**, **38**, **37**] of H. Gaussier, J. Merker, P. Nurowski, G. Sparling and A. Sukhov. The mediator between a real hypersurface $M$ and the associated ODE $\mathcal{E}(M)$ is the Segre family of $M$, which in this case is (an open subset of) the family of integral curves of $\mathcal{E}(M)$. In this paper we treat the significantly different case of a nonminimal hypersurface $M$. By establishing an injective correspondence $M\longrightarrow\mathcal{E}(M)$ between the class of all real-analytic nonminimal hypersurfaces $M\subset\mathbb{C}^2$, spherical at a generic point, and a class of second order complex ODEs with an isolated meromorphic singularity at the origin, we were able to reformulate the problems addressed in the paper in the language of analytic theory of differential equations. This gives us a powerful tool for the study of mappings and automorphisms of nonminimal hypersurfaces. The central object of the paper appears to be the nonlinear complex ODE

$$z'' = \frac{1}{w^m}(Az + B)z' + \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F), \qquad (*)$$

where the holomorphic coefficients $A(w), B(w), C(w), D(w), E(w), F(w)$ satisfy certain relations which guarantee that $(*)$ can be locally mapped into the simplest ODE $z'' = 0$ at its regular points. The latter property can be interpreted as vanishing of the Tresse differential invariants of $\mathcal{E}(M)$, or as vanishing of the Cartan curvature of $\mathcal{E}(M)$ (see the work of A. Tresse [**50**] and E. Cartan[**11**], and also V. Arnold [**2**] for a modern treatment). With the additional assumption that the hypersurface $M$ admits the rotational infinitesimal symmetry $iz\frac{\partial}{\partial z}$, the connection $M \longleftrightarrow \mathcal{E}(M)$ was studied in the earlier paper [**33**] of the authors. Remarkably, it turns out that *any such $M$ can be associated a linear ODE $z'' = \frac{B(w)}{w^m}z' + \frac{E(w)}{w^{2m}}z$,* and furthermore, Fuchsian type hypersurfaces are associated with Fuchsian ODEs. Note, however, that as examples in [**31**] show, one cannot restrict considerations to hypersurfaces with the rotational symmetry only.

The following table illustrates the relation between various geometric and ODE properties arising from the correspondence between $M$ and $\mathcal{E}(M)$.

| $M$ | $\mathcal{E}(M)$ |
|---|---|
| Nonminimal hypersurface, spherical in the complement of the complex locus $X$ | Second order complex ODE with a meromorphic singularity and vanishing Cartan-Tresse invariants at regular points |
| Nonminimal locus $X = \{w = 0\}$ | Singular point $w = 0$ |
| Segre varieties | Graphs of solutions |
| Monodromy of the associated mapping $\mathcal{F}$ | Monodromy of solutions |
| Holomorphic extension of $\mathcal{F}$ to $X$ | Meromorphic extension of solutions to $w = 0$ |
| Fuchsian type hypersurface | Fuchsian (Briot - Bouquet) type ODE |
| Automorphisms of a nonminimal hypersurface | Point symmetries of a singular ODE |

The paper is organized as follows. In Section 2 we provide some background material on CR-geometry and the analytic theory of differential equation. In Section 3 we give detailed formulations of the main results of the paper, and also formulate the necessary intermediate results. Sections 4–9 contain proofs, their organization is described at the end of Section 3.

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## 2. Preliminaries

**2.1. Segre varieties.** Let $M$ be a smooth connected real-analytic hypersurface in $\mathbb{C}^n$, $0 \in M$, and $U$ a neighbourhood of the origin where $M \cap U$ admits a real-analytic defining function $\phi(Z, \overline{Z})$, $Z = (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$. For every point $\zeta \in U$ we can associate with $M$ its so-called Segre variety in $U$ defined as

$$Q_\zeta = \{Z \in U : \phi(Z, \overline{\zeta}) = 0\}.$$

Segre varieties depend holomorphically on the variable $\overline{\zeta}$. One can find a suitable pair of neighbourhoods $U_2 = U_2^z \times U_2^w \subset \mathbb{C}^{n-1} \times \mathbb{C}$ and $U_1 \Subset U_2$ such that

$$Q_\zeta = \left\{(z, w) \in U_2^z \times U_2^w : w = h(z, \overline{\zeta})\right\}, \quad \zeta \in U_1,$$

is a closed complex analytic graph. Here $h$ is a holomorphic function. Following [**15**] we call $U_1, U_2$ a *standard pair of neighbourhoods* of the origin. The antiholomorphic $n$-parameter family of complex hypersurfaces $\{Q_\zeta\}_{\zeta \in U_1}$ is called *the Segre family of $M$ at the origin.* From the definition and the reality condition on the defining function the following basic properties of Segre varieties follow:

$$\begin{aligned} Z \in Q_\zeta \ &\Leftrightarrow\ \zeta \in Q_Z, \\ Z \in Q_Z \ &\Leftrightarrow\ Z \in M, \\ \zeta \in M \Leftrightarrow \{Z \in U_1 &: Q_\zeta = Q_Z\} \subset M. \end{aligned} \tag{2.1}$$

The fundamental role of Segre varieties for holomorphic mappings is illuminated by their invariance property: if $f : U \to U'$ is a holomorphic map sending a smooth real-analytic submanifold $M \subset U$ into another such submanifold $M' \subset U'$, and $U$ is as above, then

$$f(Z) = Z' \implies f(Q_Z) \subset Q'_{Z'}.$$

For the proofs of these and other properties of Segre varieties see, e.g., [**55**], [**14**], [**15**], [**44**], or [**4**].

In the particularly important case when $M$ is a *real hyperquadric,* i.e., when

$$M = \left\{[\zeta_0, \dots, \zeta_n] \in \mathbb{CP}^n : H(\zeta, \bar{\zeta}) = 0\right\},$$

where $H(\zeta, \bar{\zeta})$ is a nondegenerate Hermitian form in $\mathbb{C}^{n+1}$ with $k+1$ positive and $l+1$ negative eigenvalues, $k + l = n - 1$, $0 \le l \le k \le n - 1$, the Segre variety of a point $\zeta \in \mathbb{CP}^n$ is the projective hyperplane $Q_\zeta = \{\xi \in \mathbb{CP}^n : H(\xi, \bar\zeta) = 0\}$. The Segre family $\{Q_\zeta,\, \zeta \in \mathbb{CP}^n\}$ coincides in this case with the space $(\mathbb{CP}^n)^*$ of all projective hyperplanes in $\mathbb{CP}^n$.

The space of Segre varieties $\{Q_Z : Z \in U_1\}$ can be identified with a subset of $\mathbb{C}^K$ for some $K > 0$ in such a way that the so-called *Segre map* $\lambda : Z \to Q_Z$ is holomorphic (see [**14**]). For a Levi nondegenerate at a point $p$ hypersurface $M$ its Segre map is one-to-one in a neighbourhood of $p$. If $M$ contains a complex hypersurface $X$, then for any point $p \in X$ we have $Q_p = X$ and $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X$, so that the Segre map $\lambda$ sends the entire $X$ to a unique point in $\mathbb{C}^K$, and $\lambda$ is not even finite-to-one near each $p \in X$ (i.e., $M$ is *not essentially finite* at points $p \in X$). For a hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ the Segre map $\lambda'$ is a global natural one-to-one correspondence between $\mathbb{CP}^n$ and the space $(\mathbb{CP}^n)^*$.

**2.2. Defining equations for nonminimal hypersurfaces.** Let $M \subset \mathbb{C}^n$ be again a smooth real-analytic nonminimal hypersurface, containing a complex hypersurface $X \ni 0$ and Levi nondegenerate in $M \setminus X$. We choose local coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ near the origin in such a way that the complex hypersurface, contained in $M$, is given by $X = \{w = 0\}$, and $M$ is given locally by the equation

$$\operatorname{Im} w = (\operatorname{Re} w)^m \Phi(z, \bar z, \operatorname{Re} w),$$

where $\Phi(z, \bar z, \operatorname{Re} w)$ is a real-analytic function in a neighbourhood of the origin such that $\Phi(z, \bar z, 0) \not\equiv 0$, $\Phi(z, 0, \operatorname{Re} w) = \Phi(0, \bar z, \operatorname{Re} w) \equiv 0$, and $m$ is a positive integer (see [**4**], [**16**] for the existence of such coordinates). In this case $M$ is called *$m$-nonminimal,* and the integer $m$, known to be a biholomorphic invariant of $M$, is called *the nonminimality order of $M$ at* 0. We may further consider the so-called *complex defining equation* (see, e.g., [**4**]) $w = \Theta(z, \bar z, \bar w)$ of $M$ near the origin, which can be obtained by substituting $u = \frac{1}{2}(w + \bar w)$, $v = \frac{1}{2i}(w - \bar w)$ into the real defining equation and applying the holomorphic implicit function theorem. Here $\Theta = 1 + O(2)$ is a real-analytic function near the origin in $\mathbb{C}^{2n-1}$ satisfying certain reality condition. For our purposes it is convenient to use the so-called *exponential* defining equation for a nonminimal real hypersurface [**32**], [**31**]:

$$w = \bar w\, e^{i\varphi(z, \bar z,\, \bar w)},$$

where the complex-valued real-analytic function $\varphi$ in a polydisc $U \ni 0$ satisfies the conditions $\varphi(z, 0, \operatorname{Re} w) = \varphi(0, \bar z, \operatorname{Re} w) \equiv 0$ (here $m$ is the nonminimality order of $M$ at 0), $\varphi(z, \bar z,\, \bar w) = (\bar w)^{m-1} \psi(z, \bar z,\, \bar w)$ for an appropriate real-analytic function $\psi(z, \bar z,\, \bar w) \not\equiv 0$, and also the reality condition

$$\varphi(z, \bar z, w\, e^{-i\bar\varphi(\bar z, z, w)}) \equiv \bar\varphi(\bar z, z, w), \tag{2.2}$$

reflecting the fact that $M$ is a real hypersurface.

**Convention.** In what follows in this paper, for a series of the form

$$f(z_1,..,z_s)=\sum\nolimits_{k_j\in\mathbb{Z}} c_{k_1,\dots,k_s} z_1^{k_1}\cdot\ldots\cdot z_s^{k_s}$$

we denote by $\bar f(z_1,..,z_s)$ the series $\sum_{k_j\in\mathbb{Z}} \overline{c}_{k_1,\dots,k_s}\, z_1^{k_1}\cdot\ldots\cdot z_s^{k_s}$.

We introduce the following property, strengthening the $m$-nonminimality.

**Definition 2.1.** A real-analytic hypersurface $M\subset\mathbb{C}^n$, containing a complex hypersurface $X=\{w=0\}$ and Levi nondegenerate in $M\setminus X$, is called *Levi regular at the origin*, if in appropriate local coordinates near the origin the function $\varphi$ in the exponential defining equation of $M$ has the form:

$$\varphi(z,\bar z,\bar w)=(\bar w)^{m-1}(h(z,\bar z,\bar w)+\tilde\varphi(z,\bar z,\bar w)), \tag{2.3}$$

where $h(z,\bar z,\bar w)$ is a nondegenerate hermitian form in $z,\bar z$ for each $\bar w$, $m$ is the nonminimality order of $M$ at 0, $\tilde\varphi(z,0,\bar w)\equiv\tilde\varphi(0,\bar z,\bar w)\equiv 0$, and also $\tilde\varphi(z,\bar z,\bar w)=O(||z||^3)$ (here $||z||$ is the standard Euclidian norm in $\mathbb{C}^{n-1}$). Alternatively, the Levi regularity means that the power series $\frac{1}{(\bar w)^{m-1}}\varphi(z,\bar z,\bar w)|_{\bar w=0}$ has a nondegenerate hermitian part.

The following example shows that a generic nonminimal at the origin and Levi nondegenerate outside the complex locus real hypersurface does not have the Levi regularity property.

**Example 2.2.** Let $M\subset\mathbb{C}^2$ be a 2-nonminimal at the origin hypersurface of the form $\operatorname{Im} w=(\operatorname{Re} w)^4|z|^2+(\operatorname{Re} w)^2|z|^4+O(|z|^4|w|^4)$. Then it is not difficult to check that $M$ is Levi nondegenerate in $M\setminus X$, but is not Levi regular at the origin.

However, it will be shown in the next section that for $n=2$ the Levi regularity condition holds for *spherical* nonminimal hypersurfaces.

The Levi regularity condition can be naturally reformulated in terms of the real defining function $(\operatorname{Re} w)^m\Phi(z,\bar z,\operatorname{Re} w)$ above: one should require that the function $\Phi$ can be expanded as

$$\Phi(z,\bar z,\operatorname{Re} w)=H(z,\bar z,\operatorname{Re} w)+\tilde\Phi(z,\bar z,\operatorname{Re} w) \tag{2.4}$$

with $H(z,\bar z,\operatorname{Re} w)$ being a nondegenerate hermitian form in $z,\bar z$ for each $w$, $\tilde\Phi(z,0,\operatorname{Re} w)\equiv\tilde\Phi(0,\bar z,\operatorname{Re} w)\equiv 0$, and also $\tilde\Phi(z,\bar z,\operatorname{Re} w)=O(||z||^3)$. The equivalence of these definitions follows from the fact that the functions $\varphi$ and $\Phi$ from the exponential and the real defining equations respectively are related as

$$\varphi|_{M\setminus X}=\frac{1}{i}\log\frac{w}{\bar w}\bigg|_{M\setminus X}=\frac{1}{i}\log\frac{1+iu^{m-1}\Phi(z,\bar z,u)}{1-iu^{m-1}\Phi(z,\bar z,u)}$$
$$=2u^{m-1}\Phi(z,\bar z,u)+O(u^{3m-3}\Phi^3(z,\bar z,u))).$$

Here $w=u+iv$.

**2.3. Real hypersurfaces and second order differential equations.** Using the Segre family of a Levi nondegenerate real hypersurface $M \subset \mathbb{C}^n$ , one can associate to it a system of second order holomorphic PDEs with 1 dependent and $n-1$ independent variables. The corresponding remarkable construction goes back to E. Cartan [**11**],[**10**] and Segre [**46**], and was recently revisited in [**47**], [**48**], [**38**], [**18**], [**37**] (see also references therein). We describe here the procedure for the case $n = 2$, which will be relevant for our purposes. In what follows we denote the coordinates in $\mathbb{C}^2$ by $(z, w)$, and put $z = x + iy$, $w = u + iv$. Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and let $(U_1, U_2)$ be its standard pair of neighbourhoods. In this case one associates with $M$ a second order holomorphic ODE, uniquely determined by the condition that it is satisfied by the Segre family $\{Q_\zeta\}_{\zeta \in U_1}$ of $M$ in a neighbourhood of the origin where the Segre varieties are considered as graphs $w = w(z)$. More precisely, it follows from the Levi nondegeneracy of $M$ near the origin that the Segre map $\zeta \longrightarrow Q_\zeta$ is injective and also that the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U_2$, then their intersection at $q$ is transverse. Thus, $\{Q_\zeta\}_{\zeta \in U_1}$ is a 2-parameter holomorphic w.r.t. $\bar{\zeta}$ family of holomorphic curves in $U_2$ with the transversality property. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [**22**]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z, w, w')$, satisfied by the graphs $\{Q_\zeta\}_{\zeta \in U_1}$.

This procedure can be made more explicit if one considers the complex defining equation $w = \rho(z, \bar{z}, \bar{w})$ of $M$ near the origin. The Segre variety $Q_p$ of a point $p = (a, b)$ close to the origin is given by

$$w = \rho(z, \bar{a}, \bar{b}). \tag{2.5}$$

Differentiating (2.5) once, we obtain

$$w' = \rho_z(z, \bar{a}, \bar{b}). \tag{2.6}$$

Considering (2.5) and (2.6) as a holomorphic system of equations with the unknowns $\bar{a}, \bar{b}$, and applying the implicit function theorem near the origin, we get

$$\bar{a} = A(z, w, w'),\ \bar{b} = B(z, w, w').$$

The implicit function theorem here is applicable as the Jacobian of the system coincides with the Levi determinant of $M$ for $(z, w) \in M$. Differentiating (2.5) twice and plugging there the expressions for $\bar{a}, \bar{b}$ finally yields

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: H(z, w, w'). \tag{2.7}$$

Now (2.7) is the desired holomorphic second order ODE $\mathcal{E}$.

The concept of a PDE system associated with a CR-manifold can be generalized for various classes of CR-manifolds. The correspondence $M \longrightarrow \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \longrightarrow (M', 0)$ between two CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$;
(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}(M, 0)$ of $M$ at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (see, e.g., [**39**] for the details of the concept).

For the proof and applications of the properties (1) and (2) in various settings we refer to [**47**], [**48**], [**38**], [**18**], and [**37**]. Note that for a nonminimal at the origin hypersurface $M \subset \mathbb{C}^2$ there is *no* a priori way to associate with $M$ a second order ODE or even a more general PDE system near the origin. However, in Section 5 we provide a way to connect nonminimal spherical real hypersurfaces in $\mathbb{C}^2$ with a class of complex differential equations with an isolated meromorphic singularity.

**2.4. Complex linear differential equations with an isolated singularity.** Complex linear ODEs form one of the most important and geometric class of complex ODEs. We refer to [**22**], [**3**], [**9**], [**54**] and references therein for various facts and problems, concerning complex linear differential equations. A first order linear system of $n$ complex ODEs in a domain $G \subset \mathbb{C}$ (or simply a linear system in a domain $G$ in what follows) is a holomorphic ODE system $\mathcal{L}$ of the form $y'(w) = A(w)y$, where $A(w)$ is an $n \times n$ matrix-valued holomorphic in $G$ function and $y(w) = (y_1(w), ..., y_n(w))$ is an $n$-tuple of unknown functions. Solutions of $\mathcal{L}$ near a point $p \in G$ form a linear space of dimension $n$. Moreover, all the solutions $y(w)$ of $\mathcal{L}$ are defined globally in $G$ as (possibly multiple-valued) analytic functions, i.e., any germ of a solution near a point $p \in G$ of $\mathcal{L}$ extends analytically along any path $\gamma \subset G$, starting at $p$. A *fundamental system of solutions for* $\mathcal{L}$ is a matrix whose columns form some collection of $n$ linearly independent solutions of $\mathcal{L}$.

If $G$ is a punctured disc centred at 0, we call $\mathcal{L}$ *a system with an isolated singularity at* $w = 0$. An important (and sometimes even a complete) characterization of an isolated singularity is its *monodromy operator* defined as follows. If $Y(w)$ is some fundamental system of solutions of $\mathcal{L}$ in $G$ and $\gamma$ is a simple loop about the origin, then the monodromy of $Y(w)$ w.r.t. $\gamma$ is given by the right multiplication by a constant nondegenerate matrix $M$, called *the monodromy matrix.* The matrix $M$, unique up to a similarity, defines a linear operator $\mathbb{C}^n \longrightarrow \mathbb{C}^n$, which is called the monodromy operator of the singularity.

If the matrix-valued function $A(w)$ is meromorphic at the singularity $w = 0$, we call it a *meromorphic singularity.* As the solutions of $\mathcal{L}$ are holomorphic in any proper sector $S \subset G$ of a sufficiently small radius with the vertex at $w = 0$, it is important to study the behaviour of the solutions as $w \to 0$. If all solutions of $\mathcal{L}$ admit a bound $||y(w)|| \leq C|w|^b$ in any such sector (with some constants $C > 0,\ b \in \mathbb{R}$, depending possibly on the sector), then $w = 0$ is called *a regular singularity,* otherwise it is called *an irregular singularity.* In particular, in the case of the trivial monodromy the singularity is regular if and only if all the solutions of $\mathcal{L}$ are meromorphic in $G$. L. Fuchs introduced the following condition: a singular point $w = 0$ is called *Fuchsian,* if $A(w)$ is meromorphic at $w = 0$ and has a pole of order $\leq 1$ there. The Fuchsian condition turns out to be sufficient for the regularity of a singular point. Another remarkable property of a Fuchsian system is that every formal holomorphic (and even formal meromorphic) solution of a Fuchsian system is in fact convergent.

A scalar linear complex ODE of order $n$ in a domain $G \subset \mathbb{C}$ is an ODE $\mathcal{E}$ of the form

$$z^{(n)} = a_n(w)z + a_{n-1}(w)z' + ... + a_1(w)z^{(n-1)},$$

where $\{a_j(w)\}_{j=1,\dots,n}$ is a given collection of holomorphic functions in $G$ and $z(w)$ is the unknown function. By a reduction of $\mathcal{E}$ to a first order linear system (see the above references for various techniques of doing that) one can naturally transfer most of the definitions and facts, relevant to linear systems, to scalar equations of order $n$. The main difference here is contained in the appropriate definition of Fuchsian: a singular point $w = 0$ for an ODE $\mathcal{E}$ is called *Fuchsian,* if the order of poles $p_j$ of the functions $a_j(w)$ satisfy the inequalities $p_j \leq j,\ j = 1, 2, \dots, n$. The Theorem of Fuchs for $n$-th order scalar ODEs says that *a singular point of a linear $n$-th order ODE is regular if and only if it is Fuchsian.* In particular, if the monodromy of the equation is trivial, then the Fuchsian condition is equivalent to the fact that all solutions of the equation are meromorphic at the singular point $w = 0$.

Further information on the classification and behaviour of solutions for singular linear ODEs can be found in [**22**] or [**54**].

**2.5. Holomorphic vector fields and automorphisms.** We next give some preliminaries related to local automorphisms of real hypersurfaces. By a *holomorphic vector field* in a neighbourhood of the origin in $\mathbb{C}^n$ we mean a complex vector field

$$f_1(z)\frac{\partial}{\partial z_1} + ... + f_n(z)\frac{\partial}{\partial z_n},$$

where the functions $f_1(z), ..., f_n(z)$ are holomorphic in a neighbourhood of the origin. Real parts of holomorphic vector fields are precisely the real vector fields in $\mathbb{C}^n$ generating flows of local biholomorphic transformations. Let now $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface containing the origin. The *infinitesimal automorphism algebra* of $M$ at the origin (we denote it by $\mathfrak{hol}\,(M,0)$ in what follows) is the Lie algebra of germs at the origin of holomorphic vector fields $X$ such that $\operatorname{Re} X$ is tangent to $M$ at any point $p \in M$ where it is defined. If this algebra is finite-dimensional, we may assume that all of its elements are defined in the same neighbourhood of the origin. The importance of the infinitesimal automorphism algebra stems from the fact that real parts of elements of $\mathfrak{hol}\,(M,0)$ are precisely the real vector fields in $\mathbb{C}^n$ near the origin that generate real flows of local biholomorphic automorphisms of $M$ at 0.

One can also consider the *stability algebra* $\mathfrak{aut}\,(M,0)$ of $M$ at the origin. This Lie algebra consists of vector fields $X \in \mathfrak{hol}\,(M,0)$, vanishing at 0. Real parts of vector fields lying in $\mathfrak{aut}\,(M,0)$ are precisely the real vector fields in $\mathbb{C}^n$ near the origin that generate flows of local biholomorphic automorphisms of $M$ near the origin, preserving the origin. In many nondegenerate settings [**4**] this algebra is the tangent algebra to the stability group of the germ $(M,0)$.

For the compact complex manifold $\mathbb{CP}^n$, its automorphism group consists of projective transformations (given up to scaling by elements of $\mathrm{GL}(n+1,\mathbb{C})$, naturally acting in homogeneous coordinates). This Lie group is usually denoted by $\mathrm{PGL}(n+1,\mathbb{C})$. It is generated by the Lie algebra $\mathfrak{hol}(\mathbb{CP}^n)$, which is a certain algebra of quadratic vector fields in each fixed affine chart (see, for example, [**12**]). The Lie algebra $\mathfrak{hol}(\mathbb{CP}^n)$ is isomorphic to $\mathfrak{sl}(n,\mathbb{C})$ as a Lie algebra (see [**52**] for more details). For any nondegenerate hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ the algebra $\mathfrak{hol}(\mathbb{CP}^n)$ is the complexification of the infinitesimal automorphism algebra $\mathfrak{hol}(\mathcal{Q})$. It will be also important for us that the natural action of $\mathrm{PGL}(n+1,\mathbb{C})$ on $\mathfrak{hol}(\mathbb{CP}^n)$ (i.e., the natural "coordinate-change" action of biholomorphisms from $\mathrm{PGL}(n+1,\mathbb{C})$ on vector fields from $\mathfrak{hol}(\mathbb{CP}^n)$) corresponds to the adjoint action of the Lie group $\mathrm{PGL}(n+1,\mathbb{C})$ on its tangent algebra $\mathfrak{sl}(n,\mathbb{C})$. Lie algebra automorphisms corresponding to this action are sometimes called *conjugacies* or *inner automorphisms*. In the matrix realization of the above Lie groups and algebras, conjugacies are simply automorphisms given by a matrix conjugation.

**2.6. Nonminimal spherical hypersurfaces.** We give in this section more detailed formulations of the results in [**32**], which will be used in various sections of the present paper.

**Definition 2.3.** A real-analytic hypersurface $M \subset \mathbb{C}^n$, containing a complex hypersurface $X \ni 0$, is called *Segre regular in a neighbourhood $U$ of the origin,* if the Segre map $\lambda$ is locally injective in $U \setminus X$.

It is shown in [**32**, Prop 2.1] that if $M$ is Levi nondegenerate in $M \setminus X$, then one can choose a neighbourhood $U \ni 0$ in such a way that $M$ is Segre regular in $U$.

Assume now that $M$ is Levi nondegenerate in $M \setminus X$ and is Segre regular in a neighbourhood $U$. Denote by $M^+, M^-$ the two connected components of $M \setminus X$ and assume, in addition, that one of the components (say, $M^+$) is $(k, l)$-spherical (i.e., it can be locally biholomorphically mapped into a hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ with $k$ positive and $l$ negative eigenvalues, $k + l = n - 1$). The hypersurface $M$ in this case is called *pseudospherical.* Then it is proved in [**32**] that

*the second component $M^-$ is also $(k', l')$-spherical (with, possibly, $(k', l') \neq (k, l)$) and there exists an open neighbourhood $U$ of $X$ in $\mathbb{C}^n$ such that for $p \in (M \setminus X) \cap U$ any biholomorphic map $\mathcal{F}_p$ of $(M, p)$ into a $(k, l)$ - hyperquadric $\mathcal{Q}$ extends analytically along any path in $U \setminus X$ as a locally biholomorphic map into $\mathbb{CP}^n$. In particular, $\mathcal{F}_p$ extends to a possibly multiple-valued locally biholomorphic analytic mapping $\mathcal{F} : U \setminus X \longrightarrow \mathbb{CP}^n$ in the sense of Weierstrass.*

The above theorem implies the existence of a nontrivial biholomorphic invariant of a nonminimal spherical real hypersurface called *the monodromy operator.* To define it we consider a generator $\gamma$ of $\pi_1(U \setminus X)$ with $\gamma \ni p$ and consider the analytic continuation $\mathcal{F}_{\gamma,p}$ of $\mathcal{F}_p$ along $\gamma$. There exists an element $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$ such that $\mathcal{F}_{\gamma,p} = \sigma \circ \mathcal{F}_p$. It is convenient to interpret $\sigma$ as an $(n+1) \times (n+1)$-matrix, defined up to scaling, that we call *the monodromy matrix.* The monodromy matrix is defined up to similarity: namely, a replacement of the mapping $\mathcal{F}_p : (M, p) \longrightarrow \mathbb{CP}^n$ by any other mapping $\tau \circ \mathcal{F}_p : (M, p) \longrightarrow \mathbb{CP}^n$ leads to a similar monodromy matrix

$$\tilde{\sigma} = \tau \circ \sigma \circ \tau^{-1}. \tag{2.8}$$

Thus we get a well-defined linear operator $\mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$, defined up to scaling and independent of the choice of the initial mapping $\mathcal{F}_p$, the target quadric $\mathcal{Q}$ and the path $\gamma$, which is called the monodromy operator. If the analytic continuation $\mathcal{F}_{\gamma,p}$ of the initial mapping $\mathcal{F}_p$ leads to the same element $\mathcal{F}_p$, then the monodromy operator is the identity. The analytic mapping $\mathcal{F}$ in this case is a well-defined single-valued locally biholomorphic mapping $U \setminus X \longrightarrow \mathbb{CP}^n$.

## 3. Formulations of the principal results

We give in this section more detailed formulation of Theorems 1, 2, and 3, and also state some intermediate results that are of independent interest.

The first result provides the existence of prenormal coordinates for a nonminimal spherical hypersurface in $\mathbb{C}^2$. As explained earlier, prenormal coordinates do not exist for a nonminimal Levi nonflat hypersurface in general.

**Theorem 3.1.** *Let $M \subset \mathbb{C}^2$ be a real-analytic nonminimal at the origin hypersurface, and let $X$ be its complex locus. Suppose that $M \setminus X$ is Levi nondegenerate and spherical. Then in suitable local holomorphic coordinates near the origin, called prenormal coordinates, $M$ can be represented by an exponential defining equation $w = \bar{w}e^{i\varphi(z,\bar{z},\bar{w})}$ with*

$$\varphi(z,\bar{z},\bar{w}) = (\bar{w})^{m-1}\left(\pm|z|^2 + \sum_{k,l\geq 2}\varphi_{kl}(\bar{w})z^k\bar{z}^l\right), \tag{3.1}$$

*or, equivalently, by a real defining equation $\operatorname{Im} w = (\operatorname{Re} w)^m \Phi(z,\bar{z},\operatorname{Re} w)$ with*

$$\Phi(z,\bar{z},\operatorname{Re} w) = \pm|z|^2 + \sum_{k,l\geq 2}\Phi_{kl}(\operatorname{Re} w)z^k\bar{z}^l,$$

*where $\varphi_{kl}$ and $\Phi_{kl}$ are analytic functions near the origin, and $m \geq 1$ is the nonminimality order of $M$ at the origin.*

To formulate the next result we will need the following definition.

**Definition 3.2.** We denote by $\mathcal{P}_0$ the class of nonminimal smooth real-analytic hypersurfaces $M \subset \mathbb{C}^2$, containing the complex hypersurface $X = \{w = 0\}$, Levi-nondegenerate and spherical in $M \setminus X$ and given in a neighbourhood $U$ of 0 in prenormal coordinates. If in addition $U$ is a polydisc chosen in such a way that $M$ is Segre regular in $U$, we call $U$ a neighbourhood *associated with $M$.* We also call the multiple-valued mapping $\mathcal{F} : U \setminus X \longrightarrow \mathbb{CP}^2$, extending a germ $\mathcal{F}_p : (M,p) \longrightarrow (S^3,p')$ (see Section 2.6), *the mapping associated with $M$.* We call the hypersurface $M \in \mathcal{P}_0$ *positive* or *negative* depending on the sign in (3.1).

The $\mathcal{P}_0$-notation used in this paper is inherited from the analytic theory of differential equations (see Section 6 for details). Our next result establishes a fundamental connection between hypersurfaces of class $\mathcal{P}_0$ and a special class of singular complex ODEs. In what follows in the paper we denote by $\Delta_\epsilon$ a disc, centred at 0 of radius $\epsilon$, and by $\Delta_\epsilon^*$ the corresponding punctured disc.

**Theorem 3.3.** *Suppose that $M \in \mathcal{P}_0$ and $U = \Delta_\delta \times \Delta_\epsilon$ is the associated neighbourhood. Then*

*(i) There exists a second order ODE*

$$w'' = -\frac{1}{w^m}(Az + B)(w')^2 - \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F)(w')^3, \tag{3.2}$$

*where $A(w), B(w), C(w), D(w), E(w), F(w)$ are holomorphic functions in the disc $\Delta_\epsilon$ such that (3.2) is satisfied by all Segre varieties $Q_p = \{w = w_p(z)\}$, $p \in U \setminus X$, considered as graphs $w = w(z)$.*

*(ii) The ODE* (3.2) *and the complex defining function of $M$, as in* (3.1), *are related as*

$$F(w) = 2\varphi_{23}(w),\ A(w) = \pm 6i\varphi_{32}(w),\ B(w) = \pm 2i\varphi_{22}(w) - w^{m-1},$$

$$E(w) = 6\varphi_{33} \pm 2i(m-1)\varphi_{22}w^{m-1} - 8(\varphi_{22})^2 \mp 2i\varphi'_{22}w^m. \tag{3.3}$$

$$A(w) = \pm 3i\bar{F}(w),\ C(w) = -\frac{1}{9}A^2(w), \tag{3.4}$$

$$D(w) = \frac{1}{3}w^{2m}\left(\frac{A(w)}{w^m}\right)' - \frac{1}{3}A(w)B(w),$$

*where the signs are determined by the sign of $M$.*

*(iii) For a possibly smaller polydisc $U$, the Segre varieties $Q_p$ of $M$ with $p \in \Delta^*_\delta \times \Delta^*_\epsilon$, considered as graphs $z = z(w)$, satisfy the second order meromorphic ODE $\mathcal{E}(M)$, given by*

$$z'' = \frac{1}{w^m}(Az + B)z' + \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F), \tag{3.5}$$

*where $A(w), B(w), C(w), D(w), E(w), F(w)$ are the same as in* (3.2). *The correspondence $M \longrightarrow \mathcal{E}(M)$ between hypersurfaces of class $\mathcal{P}_0$ and ODEs of the form* (3.5), *satisfying* (3.4), *is injective.*

*We say that the ODE $\mathcal{E}(M)$ is associated with $M$.*

The main application of Theorem 3.3 is the possibility to reformulate questions, concerning the initial hypersurface $M$, in terms of the associated ODE $\mathcal{E}(M)$. This turns out to be a powerful tool that can be used to prove delicate facts concerning the geometry of nonminimal hypersurfaces.

We start with the applications to the problem of analytic continuation. Even though the defining equation of $M$ suggests that one should consider Segre varieties of $M$ as graphs $w = w(z)$, it appears more natural to consider them as graphs $z = z(w)$ in appropriate local coordinates. This gives characterization of nonminimal spherical hypersurfaces for which the associated mapping $\mathcal{F}$ extends holomorphically to the complex locus.

**Theorem 3.4.** *Let $M \in \mathcal{P}_0$, $U$ be the associated neighbourhood, and let $\mathcal{F}$ be the associated mapping. Then:*

*(i) There exist six (multiple-valued) analytic functions $\alpha_j(w)$ and $\beta_j(w)$, $j = 0, 1, 2$, in a punctured disc $\Delta^*_\epsilon = \{0 < |w| < \epsilon\}$ such that the mapping $\mathcal{F} : U \setminus X \longrightarrow \mathbb{CP}^2$ has the following linear w.r.t. the variable $z$ representation in homogeneous coordinates:*

$$\mathcal{F}(z, w) = (\alpha_0(w)z + \beta_0(w), \alpha_1(w)z + \beta_1(w), \alpha_2(w)z + \beta_2(w)). \tag{3.6}$$

*In particular, $\mathcal{F}$ restricted to $U^0 = \mathcal{F}^{-1}(\mathbb{C}^2)$, $U_0 \subset U \setminus X$, is a linear-fractional w.r.t. $z$ mapping $U^0 \longrightarrow \mathbb{C}^2$. Moreover, $\mathcal{F}$ extends as a* *(multiple-valued) holomorphic mapping* $\mathbb{CP}^1 \times \Delta_\epsilon^* \longrightarrow \mathbb{CP}^2$ *that is locally biholomorphic in* $\mathbb{C}^1 \times \Delta_\epsilon^*$.

*(ii) Each Segre variety* $Q_p$, $p = (a, b)$, *of* $M$ *with* $a, b \neq 0$, *considered as a subset of the strip* $\mathbb{CP}^1 \times \Delta_\epsilon^*$, *extends to a graph* $\tilde{Q}_p = \{z = h_p(w)\} \subset \mathbb{CP}^1 \times \Delta_\epsilon^*$ *of an appropriate (multiple-valued) analytic mapping* $h_p : \Delta_\epsilon^* \longrightarrow \mathbb{CP}^1$. *All functions* $h_p(w)$ *satisfy the ODE* $\mathcal{E}(M)$.

*(iii) The mapping* $\mathcal{F}$ *is single-valued if and only if for each Segre variety* $Q_p$, $p = (a, b)$, $a, b \neq 0$, *with the extension* $\tilde{Q}_p = \{z = h_p(w)\}$, *the mapping* $h_p(w)$ *is single-valued;*

*(iv) The mapping* $\mathcal{F}$ *extends to* $X$ *holomorphically if and only if for each Segre variety* $Q_p$, $p = (a, b)$, $a, b \neq 0$, *with the extension* $\tilde{Q}_p = \{z = h_p(w)\}$, *the mapping* $h_p(w)$ *is single-valued and extends to the origin holomorphically.*

Theorem 3.4 implies Theorem 2 of Introduction.

We will need the following existence theorem for singular complex ODEs, which is applicable, in particular, to the ODE $\mathcal{E}(M)$ of Theorem 3.3, provided the associated mapping is single-valued.

**Theorem 3.5.** *Consider a second order singular at the origin complex ODE* $\mathcal{E}$, *given by*

$$z'' = \frac{1}{w}P(z, w)z' + \frac{1}{w^2}Q(z, w), \tag{3.7}$$

*with holomorphic in some polydisc* $\Delta_\delta \times \Delta_\epsilon$ *functions* $P(z, w)$ *and* $Q(z, w)$. *Suppose the ODE* $\mathcal{E}$ *satisfies the following condition: if a local solution* $z = \psi(w)$ *of* $\mathcal{E}$ *near some point* $w_0 \in \Delta_\epsilon^*$ *admits an analytic continuation to an annulus* $\epsilon' < |w| < \epsilon''$, $0 < \epsilon' < \epsilon'' < \epsilon$, *then the analytic continuation is single-valued. Suppose also that there exists* $z_0 \in \Delta_\delta$ *such that* $Q(z_0, 0) = 0$. *Then the ODE* $\mathcal{E}$ *has a holomorphic at the origin solution* $z = h(w)$ *with* $h(0) = z_0$.

Combination of Theorem 3.4 with Theorem 3.5 yields the following result.

**Theorem 3.6.** *Let* $M \in \mathcal{P}_0$, $U$ *be the associated neighbourhood,* $\mathcal{F}$ *be the associated mapping, and* $m \geq 1$ *be the nonminimality order of* $M$ *at* $0$. *If* $M$ *is of Fuchsian type, then* $\mathcal{F}$ *extends to* $X$ *holomorphically if and only if it is single-valued. In particular, if* $m = 1$, *then* $\mathcal{F}$ *extends holomorphically to* $X$ *if and only if it is single-valued.*

Theorem 3.6 implies Theorem 3 stated in Introduction. Next we use the above results to study the behaviour of local automorphisms for real hypersurfaces at nonminimal points. We formulate the Dimension Conjecture, mentioned in Introduction, in two different versions.

**Dimension Conjecture (weak version).** Let $(M,0) \subset \mathbb{C}^2$ be a smooth real-analytic Levi nonflat germ. Then the following upper bound for the dimension of the stability algebra of $M$ at 0 holds:

$$\dim \mathfrak{aut}\,(M,0) \leq \dim \mathfrak{aut}(S^3,o) = 5,\ o \in S^3.$$

**Dimension Conjecture (strong version).** Let $(M,0) \subset \mathbb{C}^2$ be a smooth real-analytic Levi nonflat germ, and suppose that $M$ is not spherical at 0. Then the following upper bound for the dimension of the infinitesimal automorphism algebra of $M$ at 0 holds:

$$\dim \mathfrak{hol}\,(M,0) \leq 5.$$

As explained in Section 8 only the nonminimal case remained open for the complete proof of the strong version of the Dimension Conjecture. To treat this case, we first prove the following embedding theorem for the infinitesimal automorphism algebra of a nonminimal pseudospherical hypersurface in $\mathbb{C}^n$.

**Theorem 3.7.** *Let $M \subset \mathbb{C}^n$, $n \geq 2$, be a real-analytic nonminimal at the origin pseudospherical hypersurface. Let $\sigma$ be the monodromy operator of $M$. Then the infinitesimal automorphism algebra $\mathfrak{hol}\,(M,0)$ can be injectively embedded into the subalgebra $c = z(\sigma)\cap\mathfrak{hol}\,(\mathcal{Q})$, where $z(\sigma) \subset \mathfrak{hol}\,(\mathbb{CP}^n)$ is the centralizer of the element $\sigma \in Aut(\mathbb{CP}^n)$.*

Theorem 3.7, while being effective for hypersurfaces with nontrivial monodromy, does not give new information in the case of trivial monodromy. To treat the latter case, we use the linear-fractional representation of $\mathcal{F}$ asserted in Theorem 3.4, which gives the following.

**Theorem 3.8.** *For any hypersurface $M \in \mathcal{P}_0$ the bound $dim\,\mathfrak{hol}\,(M,0) \leq 5$ holds.*

Theorem 3.8 implies Theorem 1 in the introduction. Examples obtained in [**8**] and [**31**] show that the bound in this theorem is indeed sharp. Combined with other known results on automorphisms of real-analytic hypersurfaces in $\mathbb{C}^2$, Theorem 3.7 yields the strong version of the Dimension Conjecture.

**Theorem 3.9.** *The Strong Dimension Conjecture holds true for any smooth real-analytic hypersurface $M \ni 0$.*

In fact, we can formulate even a stronger statement. A nonminimal at the origin smooth real-analytic hypersurface $M \subset \mathbb{C}^2$ is called *a sphere blow-up,* if for some open neighbourhood $U$ of the origin there exists a holomorphic mapping $\mathcal{F} : U \longrightarrow \mathbb{C}^2$ such that $\mathcal{F}(M) \subset S^3$, $\mathcal{F}$ is locally biholomorphic in the complement $U \setminus X$ of the complex locus $X \subset M$ and $\mathcal{F}(X) = \{p'\}$ for some point $p' \in S^3$. Observe that *not* every nonminimal and spherical in $M \setminus X$ hypersurface is a sphere blow-up, as the associated mapping $\mathcal{F}$ in this case might not extend holomorphically to the complex locus $X$. We then obtain the following characterization of all real-analytic hypersurfaces with high-dimensional automorphism algebra.

**Theorem 3.10.** *Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin. Then one of the following mutually exclusive conditions hold.*

(1) *$dim\, \mathfrak{hol}\,(M,0) = \infty$ and $(M,0)$ is equivalent to the germ of the real hyperplane $\{\mathrm{Im}\, w = 0\} \subset \mathbb{C}^2$.*
(2) *$dim\, \mathfrak{hol}\,(M,0) = 8$, and $(M,0)$ is equivalent to the germ of the 3-sphere $S^3 \subset \mathbb{C}^2$.*
(3) *$dim\, \mathfrak{hol}\,(M,0) = 5$, and $(M,0)$ is a nonminimal at the origin sphere blow-up. Moreover, the Lie algebra $\mathfrak{hol}\,(M,0)$ is isomorphic to the stability algebra $\mathfrak{aut}(S^3)$ of the 3-sphere $S^3 \subset \mathbb{C}^2$.*
(4) *$dim\, \mathfrak{hol}\,(M,0) \leq 4$.*

Finally, we deduce the following description of the infinitesimal automorphism algebras of real-analytic hypersurfaces $M \subset \mathbb{C}^2$.

**Theorem 3.11.** *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, $0 \in M$, and $M$ be Levi nonflat. Then $\mathfrak{hol}\,(M,0)$ is isomorphic to a subalgebra in $\mathfrak{hol}\,(S^3) \simeq \mathfrak{su}(2,1)$, and $dim\, \mathfrak{hol}\,(M,0) \leq 5$ unless $(M,0)$ is biholomorphic to $(S^3, o)$ for $o \in S^3$.*

In the end we would like to formulate the following conjecture. It is possible to show that the Levi regularity condition, which guarantees existence of prenormal coordinates (1.2), holds on an open dense subset of the complex locus $X$ of a nonminimal Levi nonflat hypersurface. Thus one can use (1.3) to introduce the notion of Fuchsian type at a generic point $p \in X$. Following carefully the arguments in [**33**] and in the present paper, one can see that the sphericity of $M$ at a generic point does not seem to be necessary for the effect of splitting nonminimal hypersurfaces into the Fuchsian and non-Fuchsian classes (we refer again to the regularity results [**16**], [**27**] in the 1-nonminimal case). Thus we conjecture the following.

**Conjecture 3.12.** (i) The Fuchsian type is a sufficient condition for convergence of formal equivalences between nonminimal hypersurfaces. (ii) The Fuchsian type condition is sufficient for analyticity of CR-mappings between nonminimal hypersurfaces. (iii) The Fuchsian type condition is sufficient for the moderate growth, as $p \longrightarrow X$, of a mapping $\mathcal{F} : (M,p) \longrightarrow (K,p')$ from $M$ into a compact algebraic strictly pseudoconvex hypersurface $K$.

The remaining of the paper is organized as follows. In Section 4 we prove the prenormalization Theorem 3.1. Its proof is based on the globalization result [**32**] and the properties of the so-called complex Levi determinant. In Section 5 we use the associated mapping $\mathcal{F}$ to obtain a holomorphic ODE with an isolated singularity at $w = 0$, associated with $M \in \mathcal{P}_0$. We then use the existence of prenormal coordinates to obtain an associated ODE, arising from the defining function of the hypersurface. Comparing the two ODEs, we prove the meromorphic character of the associated ODE and obtain estimates for the orders of poles. We then prove in the same section Theorems 3.3 and 3.4. The crucial step is to show that the associated mapping $\mathcal{F}$ is linear-fractional in prenormal coordinates. The latter fact is proved by means of solving explicitly certain "Monge-Ampère-like" equations $I_0(z,w) = I_1(z,w) = 0$ (see Section 5 for the notations). The linear-fractional form of $\mathcal{F}$ first allows us to specify the form of the associated ODE (Theorem 3.3) and second obtain the globalization of Segre varieties and characterize the analytic continuation in terms of the behaviour of the extended Segre varieties (Theorem 3.4). As the (globalized) Segre varieties are solutions of the associated ODE $\mathcal{E}(M)$, we reformulate in Section 6 the analytic continuation problem in terms of the growth of solutions for $\mathcal{E}(M)$ as $w \longrightarrow 0$. We then reformulate the Fuchsian type condition, described in the introduction, in terms of $\mathcal{E}(M)$ and show that, under the Fuchsian type assumption, the ODE $\mathcal{E}(M)$ can be reduced by a polynomial substitution to a "Fuchs-like" ODE $\mathcal{E}^r(M)$. The latter ODE is a particular case of a Briot-Bouquet type ODE. Section 6.2 is dedicated to various examples of hypersurfaces of class $\mathcal{P}_0$ and the connections between the associated mapping, the associated ODE and the analytic continuation problem. At the end of the section we perform a crucial step in the proof of Theorem 3.6, namely, we prove that solutions of the ODE $\mathcal{E}(M)$ have a moderate growth, provided that the reduced ODE $\mathcal{E}^r(M)$ has at least one holomorphic solution, thus reducing Theorem 3.6 (and Theorem 3 from Introduction) to Theorem 3.5. In Section 7 we prove Theorem 3.5. For that one needs to prove the existence of a formal solution, which involves a simple nonresonant case and a more complicated resonant case. There we significantly use the single-valuedness of the solutions and apply the Poincaré perturbation method to show that the nonexistence of a formal solution in the resonant case leads to multiple-valuedness of some other (existing) solution, which gives a contradiction. In Section 8 we discuss the connection between the monodromy of the associated mapping $\mathcal{F}$ and the infinitesimal automorphism algebra $\mathfrak{hol}(M,0)$. This gives the proof of Theorem 3.7 and the bound $\dim \mathfrak{hol}(M,0) \leq 4$ in the case of nontrivial monodromy. We also prove the bound $\dim \mathfrak{hol}(M,0) \leq 5$ in the case when the associated mapping $\mathcal{F}$ extends to the complex locus, and thus reduce the proof of Theorem 3.8 to the case when $\mathcal{F}$ has a trivial monodromy, but does not extend to $X$. The remaining case is treated in Section 9, essentially, by proving the fact that the symmetry algebra of the associated ODE $\mathcal{E}(M)$

(at a *singular* point) has dimension at most 4. This proves Theorem 3.8 and implies Theorems 3.9, 3.10 and 3.11.

## 4. A prenormal form for a pseudospherical nonminimal hypersurface

In this section we prove the prenormalization result stated in Theorem 3.1. It is analogous to the preliminary normalization of Chern-Moser in [**12**]. Throughout this section we denote the coordinates in $\mathbb{C}^n$ by $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $w = u + iv$, and for a polydisc $U$ centred at the origin we denote by $U^z$ and $U^w$ its projections onto the $z$- and the $w$-coordinate spaces respectively. Further, we assume that $M \subset \mathbb{C}^n$ is a nonminimal real-analytic hypersurface at $0 \in M$, $X \subset M$ is the complex hypersurface through 0, $M \setminus X$ is Levi-nondegenerate and the coordinates are chosen as in Section 2.2.

As Example 2.2 shows, Theorem 3.1 fails to hold in general for nonminimal hypersurfaces, even if $M \setminus X$ is Levi nondegenerate. The proof in the spherical case is based on the study of the geometry of Segre varieties of $M$ near the complex locus $X$ and uses in essential way the result of [**32**]. The proof of the theorem is divided into several propositions.

**Definition 4.1.** Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface as above, given in a neighbourhood $U \ni 0$ by a complex defining equation $w = \rho(z, \bar z, \bar w)$. Consider $\rho(z, \bar a, \bar b)$ as a function defined in a neighbourhood of the origin in $\mathbb{C}^{2n-1}$. Then the *Levi determinant* of $M$ in $U$ is the real-analytic functional $n \times n$ determinant

$$\Delta(z,\bar a,\bar b) = \begin{vmatrix} \rho_{\bar a_1} & \cdots & \rho_{\bar a_{n-1}} & \rho_{\bar b} \\ \rho_{z_1\bar a_1} & \cdots & \rho_{z_1\bar a_{n-1}} & \rho_{z_1\bar b} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{z_{n-1}\bar a_1} & \cdots & \rho_{z_{n-1}\bar a_{n-1}} & \rho_{z_{n-1}\bar b} \end{vmatrix}.$$

For points $(z,w) \in M \cap U$, the number $\Delta(z, \bar z, \bar w)$ coincides with the determinant of the Levi form of $M$, so $\Delta(z,\bar z,\bar w) = 0$ for $(z,w) \in X$, and $\Delta(z,\bar z,\bar w) \neq 0$ for $(z,w) \in M \setminus X$, if $M \setminus X$ is Levi nondegenerate.

The Levi determinant becomes useful in determination of Levi regularity. We first prove

**Proposition 4.2** (Transversality Lemma)**.** *Suppose that $M \subset \mathbb{C}^n$ is Segre-regular in a neighbourhood $U \ni 0$ and $M$ is pseudospherical. Then if two distinct Segre varieties $Q_p, Q_q$, $p, q \in U \setminus X$ intersect at a point $s \in U \setminus X$, then their intersection is transversal.*

*Proof.* Applying [**32**, Thm 1] we conclude that there is a germ of a biholomorphic mapping $\mathcal{F}_s : (\mathbb{C}^n, s) \longrightarrow (\mathbb{CP}^n, \mathcal{F}_s(s))$ such that $\mathcal{F}_s$ sends germs of $Q_p$ and $Q_q$ at $s$ to two germs of projective hyperplanes $L_1 = \mathcal{F}_s(Q_p)$, $L_2 = \mathcal{F}_s(Q_q)$. Since $\mathcal{F}_s$ is biholomorphic, $L_1$ and $L_2$ are distinct, and so their intersection at $\mathcal{F}_s(s)$ is transverse. The same holds for the intersection $Q_p \cap Q_q$ at $s$. q.e.d.

**Proposition 4.3.** *Suppose that $M \subset \mathbb{C}^n$ is Segre-regular in a polydisc $U \ni 0$ and $M$ is pseudospherical. Then the Levi determinant $\Delta(z, \bar{a}, \bar{b})$ of $M$ is nonzero in $U^z \times U^z \times (U^w \setminus \{0\})$.*

*Proof.* Let $M$ be given by a complex defining equation $w = \rho(z, \bar{z}, \bar{w})$ and suppose that on the contrary, for some $(z^*, a^*, b^*) \in U^z \times U^z \times U^w$ with $b^* \neq 0$, the Levi determinant $\Delta(z^*, \bar{a}^*, \bar{b}^*)$ vanishes. Consider an anti-holomorphic map $\mathfrak{L}_{z^*} : U^z \times U^w \longrightarrow \mathbb{C}^n$ given by

$$\mathfrak{L}_{z^*}(a, b) = \left(\rho(z^*, \bar{a}, \bar{b}), \rho_{z_1}(z^*, \bar{a}, \bar{b}), ..., \rho_{z_{n-1}}(z^*, \bar{a}, \bar{b})\right).$$

The map $\mathfrak{L}_{z^*}$ assigns to $(a, b)$ the 1-jet of the Segre variety $Q_{\overline{(a,b)}} = \{w = \rho(z, \bar{a}, \bar{b})\}$ at the point $(z^*, \rho(z^*, \bar{a}, \bar{b}))$. Also note that $\Delta(z^*, \bar{a}^*, \bar{b}^*)$ is the Jacobian of $\mathfrak{L}_{z^*}$ at the point $(a^*, b^*)$. This implies that the map $\mathfrak{L}_{z^*}$ is degenerate at $(a^*, b^*)$, and therefore, in any small neighbourhood of $(a^*, b^*)$ there exist points $p = (a', b')$, $q = (a'', b'')$ in $U \setminus X$, $p \neq q$, such that $\mathfrak{L}_{z^*}(p) = \mathfrak{L}_{z^*}(q)$, in particular, the 1-jets of the Segre varieties $Q_p$ and $Q_q$ coincide. On the other hand, the Segre map of $M$ is locally injective, so for a sufficiently small neighbourhood of $(a^*, b^*)$ we have $Q_p \neq Q_q$. This contradicts Proposition 4.2, which proves the result.
q.e.d.

**Proposition 4.4.** *Suppose that for an $m$-nonminimal hypersurface $M \subset \mathbb{C}^2$ in a sufficiently small neighbourhood of the origin its Levi determinant $\Delta(z, \bar{a}, \bar{b}) \neq 0$ for $b \neq 0$. Then $M$ is Levi regular at $0$.*

*Proof.* Choose a neighbourhood $U \ni 0$ such that $M$ is given in $U$ by an exponential defining equation $w = \bar{w}e^{i\varphi(z,\bar{z},\bar{w})}$ and denote $\rho(z, \bar{a}, \bar{b}) := \bar{b}e^{i\varphi(z,\bar{a},\bar{b})}$. Then $\rho_{\bar{a}} = i\bar{b}\varphi_{\bar{a}}e^{i\varphi}$, $\rho_{\bar{b}} = e^{i\varphi} + i\bar{b}\varphi_{\bar{b}}e^{i\varphi}$, $\rho_{z\bar{a}} = \bar{b}e^{i\varphi}(i\varphi_{z\bar{a}} - \varphi_{\bar{a}}\varphi_z)$, $\rho_{z\bar{b}} = (i\varphi_z + i\bar{b}\varphi_{z\bar{b}} - \bar{b}\varphi_z\varphi_{\bar{b}})e^{i\varphi}$, and so

$$\Delta(z, \bar{a}, \bar{b}) = \bar{b}e^{2i\varphi}\left(-i\varphi_{z\bar{a}} + \bar{b}\varphi_{z\bar{a}}\varphi_{\bar{b}} - \bar{b}\varphi_{\bar{a}}\varphi_{z\bar{b}}\right).$$

Applying the Weierstrass Preparation Theorem and taking possibly a smaller polydisc $U$, we conclude that there exists an integer $d \geq 0$ such that

$$-i\varphi_{z\bar{a}} + \bar{b}\varphi_{z\bar{a}}\varphi_{\bar{b}} - \bar{b}\varphi_{\bar{a}}\varphi_{z\bar{b}} = (\bar{b})^d\delta(z, \bar{a}, \bar{b}),$$

where $\delta(z, \bar{a}, \bar{b})$ is holomorphic in $U^z \times U^z \times U^w$ and does not vanish there. Since $\varphi(z, \bar{a}, \bar{b}) = \bar{b}^{m-1}\psi$, $\psi = \psi_0(z, \bar{a}) + O(\bar{b})$, and $\psi_0 \not\equiv 0$ does not contain harmonic terms, we conclude that the expression $\frac{1}{\bar{b}^{m-1}}\left(-i\varphi_{z\bar{a}} + \bar{b}\,\varphi_{z\bar{a}}\,\varphi_{\bar{b}} - \bar{b}\,\varphi_{\bar{a}}\,\varphi_{z\bar{b}}\right)|_{\bar{b}=0}$ is holomorphic in $z, \bar{a}$ and does not vanish identically. Hence $d = m - 1$, and

$$\frac{1}{\bar{b}^{m-1}}\left(-i\varphi_{z\bar{a}} + \bar{b}\,\varphi_{z\bar{a}}\,\varphi_{\bar{b}} - \bar{b}\,\varphi_{\bar{a}}\,\varphi_{z\bar{b}}\right)(0,0,0) = \delta(0,0,0) \neq 0.$$

Now since $\varphi_{\bar a}(z,\bar a,\bar b)=O(|z|),\varphi_{\bar b}(z,\bar a,\bar b)=O(|z||a|)$, we conclude that $\frac{1}{(\bar b)^{m-1}}\varphi_{z\bar a}(0,0,0)\neq 0$, which is equivalent to Levi regularity. q.e.d.

Propositions 4.3 and 4.4 imply the following key

**Corollary 4.5.** *Suppose that $M\subset\mathbb{C}^2$ is an $m$-nonminimal at the origin real-analytic hypersurface, and $M\setminus X$ is Levi nondegenerate and spherical. Then $M$ is Levi regular at the origin, i.e., it can be represented in each of the forms* (2.3), (2.4).

Now, in the presence of a *leading Hermitian term* in the defining equation of a nonminimal hypersurface, we can prove Theorem 3.1 using Chern-Moser-type transformations.

*Proof of Theorem 3.1 .* First, using Corollary 4.5, we may represent $M$ in some polydisc $U$ by a defining equation $v=u^m\,\Phi(z,\bar z,u)$, where $\Phi(z,\bar z,u)$ is given as in (2.4). In the proof we denote by $O_{22}$ a power series in $z$, $\bar z$, and $u$ containing only monomials $z^k\bar z^l u^j$, $k,l\geq 2$, $j\geq 0$. We consider the expansion

$$\tilde\Phi(z,\bar z,u)=z\lambda(\bar z,u)+\bar z\bar\lambda(z,u)+O_{22},$$

and $H(z,\bar z,u)=\alpha(u)|z|^2$, where $\alpha(u)\neq 0$ in $U^w$. Define the function $f(z,w)=\frac{\bar\lambda(z,w)}{\alpha(w)}$. Note that for $(z,w)\in M$, we have $\bar w=u-iu^m\,O(|z|^2)$, so $\left.\frac{\alpha(u)}{\alpha(\bar w)}\right|_M=1+O(|z|^2)$. Therefore,

$$\begin{aligned}H(z+f(z,w),\bar z+\bar f(\bar z,\bar w),u)|_M&=\\(H(z,\bar z,u)+z\lambda(\bar z,\bar w)+\bar z\bar\lambda(z,w))|_M+O_{22}&=\\H(z,\bar z,u)+z\lambda(\bar z,u)+\bar z\bar\lambda(z,u)+O_{22}.&\end{aligned}$$

From this it follows that the transformation

$$z^*=z+f(z,w),\;\;w^*=w$$

maps $M$ onto a hypersurface $M^*$ given by

$$v^*=(u^*)^m\left(H(z^*,\bar z^*,u^*)+\sum_{k,l\geq 2}\varphi^*_{kl}(u^*)(z^*)^k(\bar z^*)^l\right). \tag{4.1}$$

Finally, to make $H$ independent of $u$ for $M$ given by (4.1), we drop the asterisks for the sake of simplicity and set $H(z,\bar z,u)=\alpha(u)|z|^2$ with $\alpha(u)\neq 0$. Since $\alpha(u)$ is real-valued, we may assume first that $\alpha(u)>0$. The transformation

$$z^*=z\sqrt{\alpha(w)},\;\;w^*=w,$$

where the root is chosen to be positive for the positive argument, maps $M$ onto the hypersurface of the form (1.2). This follows from $\left|z\sqrt{\alpha(w)}\right|^2=H(z,\bar z,u)+O_{22}$ whenever $(z,w)\in M$. The proof for $\alpha(u)<0$ is analogous. q.e.d.

## 5. Ordinary differential equation associated with a nonminimal spherical hypersurface

In this section we prove Theorem 3.3, which describes a (singular) second order ODE associated with the real hypersurface $M$. We also prove Theorem 3.4, which allows us to reduce the study of the associated mapping $\mathcal{F}$ to the study of solutions for the associated ODE. As explained in Section 2, in general, a nonminimal real hypersurface does not admit a second order ODE associated with it. However, such ODE always exists in the spherical case. The proof of this crucially depends on the global properties of the mapping $\mathcal{F}$ associated with $M$.

**5.1. Existence of an associated singular ODE.** In what follows we assume that $M$ is a hypersurface of class $\mathcal{P}_0$, $U$ is the associated neighbourhood, and $\mathcal{F}$ is the associated mapping. We start by introducing the *regular set* $U^0 = \mathcal{F}^{-1}(\mathbb{C}^2) \subset U \setminus X$ and the *exceptional set* $E = (U \setminus X) \setminus U^0 = \mathcal{F}^{-1}(\mathbb{CP}^2 \setminus \mathbb{C}^2)$. The set $E$ is the pre-image of the projective line $\mathbb{CP}^2 \setminus \mathbb{C}^2$, and since each element of $\mathcal{F}$ at a point $p \in U \setminus X$ is biholomorphic in a sufficiently small polydisc, $E$ is a locally countable union of one-dimensional locally complex-analytic sets in $U \setminus X$. This implies that $E$ has Hausdorff dimension 2, so that $U^0$ is an open, connected and dense subset in $U \setminus X$, see, e.g, [**13**]. We first study the behaviour of $\mathcal{F}$ on the regular set.

Fix a point $p \in U^0$ and a biholomorphic element $\mathcal{F}_p$ of $\mathcal{F}$ at $p$, defined in a sufficiently small polydisc $U_p \subset U^0$. We claim that in $U_p$ there exists a second order ODE that is satisfied by all Segre varieties of $M$ that have nonempty intersection with $U_p$.

To prove the claim we write $\mathcal{F}_p = (f, g)$, as the components of $\mathcal{F}$ are well-defined in $U^0$. For some point $s \in Q_p$ there exists a polydisc $U_s \subset U \setminus X$ such that $\cup_{q \in U_s} Q_q$ contains a neighbourhood of $p$. By shrinking $U_p$, we may assume that this neighbourhood is $U_p$. The $\mathfrak{Q}$-Segre property of $\mathcal{F}$ (see [**32**, Prop. 4.1]) implies that $\mathcal{F}_p$ sends open pieces $Q_q \cap U_p$ of Segre varieties to affine complex lines $\Pi_q \subset \mathbb{C}^2$. For a fixed $q \in U_s$, assume that $\Pi_q$ is given by

$$z^*\lambda + w^*\mu = 1 \tag{5.1}$$

for some $\lambda, \mu \in \mathbb{C}$, with $\mu \neq 0$. Setting $(z^*, w^*) = (f, g)$ we see that the set $Q_q \cap U_p$, considered as a graph $w = w_q(z), z \in U_p^z$, satisfies the equation:

$$f(z, w_q(z))\lambda + g(z, w_q(z))\mu = 1. \tag{5.2}$$

Differentiation of (5.2) once w.r.t. $z$ yields

$$\begin{aligned} &f_z(z, w_q(z))\lambda + g_z(z, w_q(z))\mu + f_w(z, w_q(z))w_q'(z)\lambda \\ &+g_w(z, w_q(z))w_q'(z)\mu = 0. \end{aligned} \tag{5.3}$$

Consider (5.2) and (5.3) as a system of linear equations w.r.t. $\lambda$ and $\mu$. This system correctly defines a map $(z,q) \to (\lambda,\mu)$. Indeed, suppose that for some $(z^0,q^0)$ there exist more than one solution $(\lambda,\mu)$ of the system (5.2), (5.3). Then (5.2) implies that for all solutions $(\lambda,\mu)$ the corresponding complex lines (5.1) pass through the point $\mathcal{F}(z^0, w_{q^0}(z^0))$, while (5.3) implies that the line $D\mathcal{F}(T_{(z^0,w_{q^0}(z^0))}Q_{q^0})$ is tangent to (5.1). But since $D\mathcal{F} \neq 0$, it follows that there exists only one such pair $(\lambda,\mu)$. By solving the system (5.2), (5.3) we may express $\lambda$ and $\mu$ as functions of $(z,q)$. By the invariance of Segre varieties, these are, in fact, functions of $q$ only.

Differentiating (5.2) twice yields (we omit the arguments for simplicity of the formula)

$$\begin{aligned} &w''(\lambda f_w + \mu g_w) + (w')^2(\lambda f_{ww} + \mu g_{ww}) \\ &+w'(2\lambda f_{zw} + 2\mu g_{zw}) + (\lambda f_{zz} + \mu g_{zz}) = 0. \end{aligned} \tag{5.4}$$

Now, substitution of $\lambda$ and $\mu$ with solutions of the system, gives

$$\begin{aligned} w''(f_w g_z - f_z g_w) = (f_z + f_w w')(g_{zz} + 2g_{zw}w' + g_{ww}(w')^2) \\ -(g_z + g_w w')(f_{zz} + 2f_{zw}w' + f_{ww}(w')^2). \end{aligned}$$

Since $\mathcal{F}_p$ is biholomorphic in $U_p$, the Jacobian $J = f_w g_z - f_z g_w$ is nonzero in $U_p$, and we obtain

$$w'' = I_0 + I_1 w' + I_2(w')^2 + I_3(w')^3, \tag{5.5}$$

where

$$\begin{aligned} I_0 &= \tfrac{1}{f_w g_z - g_w f_z}\left(f_z g_{zz} - g_z f_{zz}\right), \\ I_1 &= \tfrac{1}{f_w g_z - g_w f_z}\left(f_w g_{zz} - g_w f_{zz} + 2 f_z g_{zw} - 2 g_z f_{zw}\right), \\ I_2 &= \tfrac{1}{f_w g_z - g_w f_z}\left(f_z g_{ww} - g_z f_{ww} + 2 f_w g_{zw} - 2 g_w f_{zw}\right), \\ I_3 &= \tfrac{1}{f_w g_z - g_w f_z}\left(f_w g_{ww} - g_w f_{ww}\right). \end{aligned} \tag{5.6}$$

Furthermore, (5.5) is satisfied by $Q_q \cap U_p$ for all $q \in U \setminus X$ with $Q_q \cap U_p \neq \emptyset$, not just for $q \in U_s$. To see this, observe that there exists a pair of polydiscs $U_1 \Subset U_2 \Subset U_p$ with the property that if $Q_q \cap U_1 \neq \varnothing$, then $Q_q \cap U_2$ is a graph $w = w_q(z)$ over $U_2^z$. We shrink $U_p$ to $U_1$ and consider $Q_q$ as graphs in $U_2$. Then the assertion follows from the analytic dependence of $Q_q$ on $q$, and the fact that the set $\{q : Q_q \cap U_p \neq \emptyset\}$ coincides with $\cup Q_r, r \in U_p$, and hence is open and connected. This proves our claim.

Since $\mathcal{F}_p$ extends analytically along any path in $U^0$, so do the four analytic elements $I_0(z,w), I_1(z,w), I_2(z,w), I_3(z,w)$. On the other hand, equation (5.5) is independent of the choice of the germ of $\mathcal{F}$ at $p$. This can be argued as follows: from the previous discussion we may conclude that $\{Q_q \cap U_p,\ q \in U_s\}$ is an anti-holomorphic 2-parameter family of holomorphic curves in $U_p$. Then this family has the transversality property, i.e., the map $(z,\alpha,\beta) \to (z, w_{(\alpha,\beta)}(z), w'_{(\alpha,\beta)}(z))$ is injective, and thus there exists a unique second order ODE $w'' = \theta(z,w,w')$ satisfied by the family $\{Q_q \cap U_p,\ q \in U_s\}$ (see Section 2.3). From this we conclude that the ODE (5.5) is unique, i.e., is independent of the choice of $\mathcal{F}_p$.

From the uniqueness of (5.5) we conclude that the four functions $I_0$, $I_1$, $I_2$, $I_3$ are holomorphic in all of $U^0$, in particular, single-valued. For the same reason the replacement of the mapping $\mathcal{F}$ by a mapping $\sigma \circ \mathcal{F}$, $\sigma \in \mathfrak{hol}(\mathbb{CP}^2)$, does not change the expressions $I_0$, $I_1$, $I_2$, $I_3$ in a neighbourhood of $p$, provided $\sigma \circ \mathcal{F}_p$ is still a mapping to the affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$.

Take now a point $p \in E$ and replace $\mathcal{F}$ by the mapping $\tilde{\mathcal{F}} = \sigma \circ \mathcal{F}$ such that $\sigma \in \mathfrak{hol}(\mathbb{CP}^2)$ and $\tilde{\mathcal{F}}_p = \sigma \circ \mathcal{F}_p \subset \mathbb{C}^2$ maps $U_p$ into $\mathbb{C}^2$. Then the regular set $U^0$ is replaced by an open dense set $\tilde{U}^0$ and using the map $\tilde{\mathcal{F}}$ we obtain a second order ODE in a neighbourhood of $p$ with the properties analogous to those of (5.5). This shows that $I_0, I_1, I_2, I_3$ extend holomorphically to $E$.

Finally, since $I_0$, $I_1$, $I_2$, $I_3$ are holomorphic in $U \setminus X$, we conclude that (5.5) is satisfied by all *entire* (i.e., in all of $U \setminus X$) Segre varieties $Q_q$ for $q \in U \setminus X$. We summarize our arguments in the following key

**Proposition 5.1.** *In the assumptions of Theorem 3.3, there exist four holomorphic in $U \setminus X$ functions $I_0(z,w)$, $I_1(z,w)$, $I_2(z,w)$, $I_3(z,w)$ such that the differential equation* (5.5) *is satisfied by the defining function $w_q(z)$ of each of the Segre varieties $Q_q$, $q \in U \setminus X$, considered as graphs $w = w_q(z)$. In each neighbourhood $U_p$, $p \in U^0$, and for any element $\mathcal{F}_p$ of $\mathcal{F}$ with $\mathcal{F}_p(U_p) \subset \mathbb{C}^2$ that has components $(f,g)$ as a map $U_p \longrightarrow \mathbb{C}^2$, the four functions $I_0, I_1, I_2, I_3$ are given by* (5.6). *The expressions in* (5.6) *are invariant under the exterior action of elements $\sigma \in Aut(\mathbb{CP}^2)$ with $\sigma(\mathcal{F}_p(U_p)) \subset \mathbb{C}^2$. At points $p \in E$ the four expressions $I_0, I_1, I_2, I_3$ can be computed by formulas* (5.6) *applied to $\sigma \circ \mathcal{F}_p$ with $\sigma(\mathcal{F}_p(U_p)) \subset \mathbb{C}^2$.*

We now determine the behaviour of $I_0, I_1, I_2, I_3$ near the complex locus $X$, using smoothness of $M$ given in prenormalized form (3.1).

**Proposition 5.2.** *In the assumptions of Theorem* 3.3 *one has $I_0 = I_1 \equiv 0$. Furthermore, the functions $w^m I_2(z,2)$, $w^{2m} I_3(z,w)$ extend to $X$ holomorphically, i.e., $I_2$ has the pole of order $\leq m$ w.r.t $w$ at $w = 0$ and $I_3$ has the pole of order $\leq 2m$ w.r.t $w$ at $w = 0$.*

*Proof.* We find a relationship between the defining function $\varphi$ as in (3.1) and (5.5). Assume first that $M$ is positive. Let $q = (a,b) \in U \setminus X$ so that $b \neq 0$. Then $Q_q$ is given by

$$w = w(z) = \bar{b} e^{i\varphi(z,\bar{a},\bar{b})} = \bar{b} + i\bar{b}^m z\bar{a} + O(z^2\bar{a}^2\bar{b}^m). \tag{5.7}$$

Differentiation of (5.7) w.r.t. $z$ yields

$$w' = i\bar{b}e^{i\varphi(z,\bar{a},\bar{b})}\varphi_z(z,\bar{a},\bar{b})$$

$$= (i\bar{b} - z\bar{a}\bar{b}^m + O(z^2\bar{a}^2\bar{b}^m))(\bar{a}\bar{b}^{m-1} + O(z\bar{a}^2\bar{b}^{m-1})) = i\bar{a}\bar{b}^m + O(z\bar{a}^2\bar{b}^m).$$

Now set $\zeta := \frac{w'}{w^m}$; this is well-defined because $w(z) \neq 0$ for a Segre variety $Q_q$, $q \in U \setminus X$. Then, combining the last equalities, we obtain

$$\zeta = i\bar{a} + O(z\bar{a}^2\bar{b}^m). \tag{5.8}$$

Equation (5.8) shows that choosing possibly a smaller initial neighbourhood $U$ of the origin we can apply the implicit function theorem for the system (5.7), (5.8) near the origin to get

(5.9)

$$\bar{a} = P(z,w,\zeta) = -i\zeta + O(z\zeta^2 w^m),\ \bar{b} = Q(z,w,\zeta) = w + O(z\zeta w^m).$$

Differentiating (5.7) twice w.r.t. $z$ and plugging (5.9) into the result we conclude that for each point $(z,w) \in Q_q$ the values $z, w, w', w''$ are related by

$$\begin{aligned} w'' &= O(P(z,w,\zeta)^2 \cdot Q(z,w,\zeta)^m) \\ &= \sum_{j\geq 0, k\geq 2, l\geq m} h_{jkl} z^j \left(\frac{w'}{w^m}\right)^k w^l := \Phi(z,w,w'). \end{aligned} \tag{5.10}$$

We note that the values $(z,w,w')$ in (5.10) belong to some open domain $\Omega \subset \mathbb{C}^3$ (to see the openness we argue as in the proof of Proposition 5.1 and consider the locally biholomorphic mapping $\chi:\ (z,q) \longrightarrow (z, w_q(z), w_q'(z))$, $q \in U \setminus X$, $z \in U^z$; then simply $\Omega = \chi(U \setminus X)$). The series (5.10) converges in $\Omega$ uniformly on compact subsets. From (3.1) we get $\Phi(z,w,w') \equiv I_0(z,w) + I_1(z,w)w' + I_2(z,w)(w')^2 + I_3(z,w)(w')^3$ (as the uniqueness implies). On the other hand, considering the biholomorphic in $\Omega$ mapping $\psi:\ (z,w,w') \longrightarrow (z,w,\frac{w'}{w^m}) = (z,w,\zeta)$ we obtain a domain $\tilde{\Omega} = \psi(\Omega) \subset \mathbb{C}^3$ and may consider the holomorphic in $\tilde{\Omega}$ function $H(z,w,\zeta) := \Phi(z,w,\zeta w^m)$. Then $H(z,w,\zeta)$ is given in $\tilde{\Omega}$ by the power series

$$\sum_{j\geq 0, k\geq 2, l\geq m} h_{jkl} z^j \zeta^k w^l. \tag{5.11}$$

This implies that there exists a polydisc $V \subset \mathbb{C}^3$, centred at 0, such that $V \cap \tilde{\Omega} \neq \emptyset$ and the power series (5.11) converges in $V$. Then in $V \cap \tilde{\Omega}$ we have

$$H(z,w,\zeta) = \Phi(z,w,\zeta w^m)$$

$$= I_0(z,w) + I_1(z,w)\zeta w^m + I_2(z,w)\zeta^2 w^{2m} + I_3(z,w)\zeta^3 w^{3m}.$$

Comparing with (5.11) we finally obtain that

$$I_0(z,w) \equiv 0,\ I_1(z,w) \equiv 0,$$
$$I_2(z,w)w^{2m} = \sum_{j\geq 0, l\geq m} h_{j2l} z^j w^l,\ I_3(z,w)w^{3m} = \sum_{j\geq 0, l\geq m} h_{j3l} z^j w^l.$$

This proves the proposition in the positive case. The negative case is analogous. q.e.d.

### 5.2. Proof of Theorem 3.3(i) and representation (3.6).

*Proof of representation* (3.6). Choose a point $p \in U^0$ and an element $\mathcal{F}_p = (f,g) : U_p \longrightarrow \mathbb{C}^2$ of $F$ in a polydisc $U_p = U^z \times U_p^w \subset U_p^0$ centred at $p$. We consider (5.6) and use the two identities $I_0(z,w) \equiv 0$ and $I_1(z,w) \equiv 0$ proved in Proposition 5.2. The first one gives $f_z g_{zz} - g_z f_{zz} = 0$, so that $\left(\frac{g_z}{f_z}\right)_z = 0$ assuming $f_z \neq 0$, while the second implies $g_z = \lambda(w) f_z$, so that

$$g(z,w) = \lambda(w) f(z,w) + \mu(w) \tag{5.12}$$

for some $\lambda(w), \mu(w)$ holomorphic in $U_p^w$. Plugging (5.12) into $I_1(z,w) \equiv 0$ yields

$$-\lambda' f f_{zz} - \mu' f_{zz} + 2\lambda'(f_z)^2 = 0. \tag{5.13}$$

By the implicit function theorem, using the condition $f_z \neq 0$, there exists a function $P(\zeta, w)$, holomorphic in $\{f(U_p)\} \times U_p^w$, such that $f_z = P(f,w)$. Then $f_{zz}(z,w) = P(f(z,w),w) P_\zeta(f(z,w),w)$, which can be rewritten in a simple form $f_{zz} = PP_f$. Substituting this into (5.13) gives

$$-\lambda' f P P_f - \mu' P P_f + 2\lambda' P^2 = 0.$$

This can be considered, for each fixed $w$, as a first-order elementary differential equation with the independent variable $f$ and the dependent variable $P$. Separation of variables gives $\frac{P_f}{P} = \frac{2\lambda'}{\mu' + \lambda' f}$. After integration we conclude that $P = \rho(w)(\mu'(w) + \lambda'(w) f)^2$ for some function $\rho(w)$ holomorphic in $U^w$. So finally we obtain another first-order elementary ODE

$$f_z = \rho(\mu' + \lambda' f)^2$$

with the independent variable $z$ and the dependent variable $f$. Separating variables and integrating, we get $-\frac{1/\lambda'}{\mu' + \lambda' f} = \rho z + \nu$ for an appropriate holomorphic function $\nu(w)$. The latter equality implies that $f$ is linear-fractional w.r.t. $z$ in $U_p$. Changing the notation and using (5.12), we conclude that

$$f(z,w) = \frac{\alpha_1(w)z + \beta_1(w)}{\alpha_0(w)z + \beta_0(w)},\ g(z,w) = \frac{\alpha_2(w)z + \beta_2(w)}{\alpha_0(w)z + \beta_0(w)} \tag{5.14}$$

for appropriate holomorphic functions $\alpha_0(w), ..., \beta_2(w)$ in $U^w$, which is equivalent to (3.6) restricted to the polydisc $U_p$. The collection $\alpha_0(w), ..., \beta_2(w)$ is defined uniquely up to scaling by a function $h(w)$, holomorphic and nonzero in $U^w$. Returning to the assumption $f_z \neq 0$, observe that the Jacobian $f_w g_z - g_w f_z$ is nonzero in $U_p$, so that interchanging, if necessary, $f$ and $g$, we may still assume that the condition $f_z \neq 0$ holds true in a sufficiently small polydisc centred at $p$. Thus, (5.14) holds in the general case as well. Note also that the form (5.14) is invariant under projective transformations in the image-space $\mathbb{CP}^2$. This means that after replacing $F$ by an appropriate composition, equation (5.14) holds for a small neighbourhood of an arbitrary point $p \in U \setminus X$.

Consider now two polydiscs $U_p$ and $U_q$, $p, q \in U \setminus X$, with $U_p \cap U_q \neq \emptyset$, and two elements $F_p, F_q$ there such that $F_p = F_q$ in $U_p \cap U_q$. Given the representations (5.14) in both polydiscs, we may solve a simple multiplicative Cousin problem to show that the collections $\alpha_0(w), ..., \beta_2(w)$ can be scaled in such a way that they coincide in $U_p \cap U_q$. This means that each fixed collection $\alpha_0(w), ..., \beta_2(w)$ in a polydisc $U_p$ can be extended analytically along an arbitrary path in $U \setminus X$, starting at $p$, because the mapping $F$ does, and this proves the representation (3.6). q.e.d.

*Proof of Theorem 3.1(i).* To prove (3.2) we find the functions $I_2(z, w)$, $I_3(z, w)$, using the linear-fractional representation (3.6). As the functions $w^m I_2(z, w)$, $w^{2m} I_3(z, w)$ are holomorphic in the entire neighbourhood $U$ (from Proposition 5.2), for the proof of (4.1) it suffices to show that $I_2$ is linear and $I_3$ is cubic w.r.t. the variable $z$. In fact, we can do that in a neighbourhood $U_p \subset U^0$ of a point $p \in U^0$. We first suppose $\alpha_0(w) \not\equiv 0$ and change the form of the representation (3.6), rewriting it in $U_p$ for a fixed element $F_p$ of $F$ as

$$f(z, w) = \frac{\alpha}{z + \delta} + \beta, \; g(z, w) = \frac{a}{z + \delta} + b, \tag{5.15}$$

where $\alpha(w), \beta(w), \delta(w), a(w), b(w)$ are meromorphic in $U_p^w$. Then $I_1 \equiv 0$ and (5.6) imply

$$\beta' a - b' \alpha \equiv 0, \tag{5.16}$$

which after differentiation gives $\beta'' a - b'' \alpha + \beta' a' - b' \alpha' = 0$. Straightforward computations using (5.16) give

$$J = f_w g_z - g_w f_z = \frac{a' \alpha - \alpha' a}{(z + \delta)^3}, \tag{5.17}$$

$$I_2(z,w) = \left[\frac{a\alpha'' - \alpha a''}{a'\alpha - \alpha' a}\right] + 3\left[\frac{b'\alpha' - \beta' a'}{a'\alpha - \alpha' a}\right](z+\delta), \tag{5.18}$$

$$\begin{aligned} I_3(z,w) = &\left[\delta'' + \delta'\frac{a\alpha'' - \alpha a''}{a'\alpha - \alpha' a}\right] \\ &+ \left[\frac{a''\alpha' - \alpha'' a'}{a'\alpha - \alpha' a} + 3\delta'\frac{b'\alpha' - \beta' a'}{a'\alpha - \alpha' a}\right](z+\delta) + \\ &+ \left[\frac{\beta' a'' - b'\alpha'' + \alpha' b'' - a'\beta''}{a'\alpha - \alpha' a}\right](z+\delta)^2 + \left[\frac{\beta' b'' - b'\beta''}{a'\alpha - \alpha' a}\right](z+\delta)^3. \end{aligned} \tag{5.19}$$

The identities (5.18) and (5.19) demonstrate the desired polynomial dependence. Suppose now that $\alpha_0 \equiv 0$. Since $f$ is a local biholomorphism, $\alpha_1$ and $\alpha_2$ cannot be both identically zero. Thus, after relabelling the functions, we return to the previous case. This completely proves statement (i) of Theorem 3.3. q.e.d.

**5.3. Proof of Theorem 3.4.** In what follows, by $\mathcal{M}(0)$ (resp. $\mathcal{O}(0)$) we denote the space of germs at the origin of meromorphic (resp. holomorphic) functions in $w \in \mathbb{C}$.

*Proof of Theorem 3.4.* Note that (3.6) is proved in Section 5.2. To complete the proof of statement (i) we need to show that $\mathcal{F}$ extends from $U \setminus X = \{|z| < \delta\} \times \Delta^*_\epsilon$ to $\mathbb{CP}^1 \times \Delta^*_\epsilon$ analytically and the restriction of $\mathcal{F}$ to $\mathbb{C} \times \Delta^*_\epsilon$ is locally biholomorphic in $\Delta^*_\epsilon$. Using representation (3.6), we extend $\mathcal{F}$ as

$$\begin{aligned} \tilde{\mathcal{F}}(z,w) := (\alpha_0(w)\mathbf{z} + \beta_0(w)\mathbf{t}, \alpha_1(w)\mathbf{z} \\ + \beta_1(w)\mathbf{t}, \alpha_2(w)\mathbf{z} + \beta_2(w)\mathbf{t}) \in \mathbb{CP}^2 \end{aligned} \tag{5.20}$$

(we fixed here a germ of each of the functions $\alpha_0, ..., \beta_2$ and denote by $(\mathbf{z}, \mathbf{t})$ the homogeneous coordinates in $\mathbb{CP}^1$). To prove that $\tilde{\mathcal{F}}$ is, in fact, analytic, we need to show that the 3 expressions $\alpha_0(w)\mathbf{z} + \beta_0(w)\mathbf{t}$, $\alpha_1(w)\mathbf{z} + \beta_1(w)\mathbf{t}$, and $\alpha_2(w)\mathbf{z} + \beta_2(w)\mathbf{t}$ cannot vanish for $(\mathbf{z}, \mathbf{t}) \neq (0,0)$, $0 < |w| < \epsilon$.

We first observe that $\alpha_0(w^*) = \alpha_1(w^*) = \alpha_2(w^*) = 0$ is not possible for any fixed $w^*$, $0 < |w^*| < \epsilon$, since otherwise, by (3.6), $\mathcal{F}(z, w^*)$ is independent of $z$, but $\mathcal{F}$ is biholomorphic in $U \setminus X$. Assume now that for some $\mathbf{z}^*, \mathbf{t}^*, w^*$ one has

$$\alpha_0(w^*)\mathbf{z}^* + \beta_0(w^*)\mathbf{t}^* = \alpha_1(w^*)\mathbf{z}^* + \beta_1(w^*)\mathbf{t}^* = \alpha_2(w^*)\mathbf{z}^* + \beta_2(w^*)\mathbf{t}^* = 0.$$

Suppose first $\mathbf{t}^* \neq 0$. Then $\beta_j(w^*) = -\frac{\mathbf{z}^*}{\mathbf{t}^*}\alpha_j$, $j = 0,1,2$. Then for $z \neq z^*/t^*$,

$$\begin{aligned} F(z,w^*) &= \left[\alpha_0(w^*)\left(z - \frac{z^*}{t^*}\right),\ \alpha_1(w^*)\left(z - \frac{z^*}{t^*}\right),\ \alpha_2(w^*)\left(z - \frac{z^*}{t^*}\right)\right] \\ &= [\alpha_0(w^*),\ \alpha_1(w^*),\ \alpha_2(w^*)]. \end{aligned}$$

This means that the line $\{(z, w^*)\}$ is mapped into a point, which is a contradiction. Similarly, if $t^* = 0$ then, in view of $z^* \neq 0$, we conclude that $\alpha_0(w^*) = \alpha_1(w^*) = \alpha_2(w^*) = 0$ which is not possible by the above argument.

To show that all elements of $\tilde{\mathcal{F}}$, i.e., local maps obtained by analytic continuation, at points lying in $\mathbb{C} \times \Delta^*_\epsilon$ are locally biholomorphic, we fix $p = (z^*, w^*) \in \mathbb{C} \times \Delta^*_\epsilon$, choose a polydisc $U_p \subset \mathbb{C} \times \Delta^*_\epsilon$ and replace $\tilde{\mathcal{F}}$, if necessary, with $\sigma \circ \tilde{\mathcal{F}}$ for an appropriate $\sigma \in \mathrm{Aut}(\mathbb{CP}^2)$ in order to have $\tilde{\mathcal{F}}(U_p) \subset \mathbb{C}^2$. Note that $\tilde{\mathcal{F}}_p$ admits a single-valued extension to $U(w^*) \times \mathbb{C}$ for some disc $U(w^*)$, centred at $w^*$, using (5.20). Then (5.15) and (5.17) show that $\mathcal{F}_p$ is biholomorphic near $p$, unless $(a'\alpha - \alpha' a)(w^*) = 0$. Choosing now $z$ such that $|z| < \delta$ and $\mathcal{F}(z, w^*) \in \mathbb{C}^2$, which is possible since $\mathcal{F}_p$ maps an open piece of the line $\mathbb{C} \times \{w^*\}$ to $\mathbb{C}^2$, we conclude that $\mathcal{F}_p$ is not biholomorphic at $(a, w^*) \in U \setminus X$. This is a contradiction, and statement (i) is proved.

In order to prove (ii), we first fix $p = (p_1, p_2) \in U$, $p_1, p_2 \neq 0$, and consider $Q_p$ as the graph $w = \theta_p(z)$. Expanding as in (5.7), we get $\theta_p(z) = \bar{p}_2 + i\bar{p}_1\bar{p}_2^m z + O(z^2\bar{p}_1^2\bar{p}_2^m)$. Choosing now a possibly smaller polydisc $U$, we may assume $\theta_p(z)$ is injective in $\{|z| < \delta\}$, where $\delta$ is independent of $p$. Indeed, $\theta_p(z) = \theta_p(z^*)$ implies from (5.7) that $(z - z^*)[1 + O(|p_1|)] = 0$, and that implies injectivity of $\theta_p(z)$ for all $p$. We may then consider the inverse holomorphic function $z = \psi(w)$ in some domain $\Delta_p \subset \Delta^*_\epsilon$. The graphs $w = \theta(z)$ and $z = \psi(w)$ both coincide with $Q_p$. As $Q_p$ is simply-connected, we may consider a single-valued restriction $\mathcal{F}_p$ of $\mathcal{F}$ to a simply-connected neighbourhood $V$ of $Q_p$. Then $\mathcal{F}_p(Q_p)$ is contained in a projective line $\lambda_0\xi_0 + \lambda_1\xi_1 + \lambda_2\xi_2 = 0$. From (5.20), the substitution

$$(\xi_0, \xi_1, \xi_2) = (\alpha_0(w)\mathbf{z} + \beta_0(w)\mathbf{t}, \alpha_1(w)\mathbf{z} + \beta_1(w)\mathbf{t}, \alpha_2(w)\mathbf{z} + \beta_2(w)\mathbf{t})$$

into the equation of the projective line shows that $Q_p$ is a subset of a bigger set

$$\{P(w)\mathbf{z} + Q(w)\mathbf{t} = 0\} \subset \mathbb{CP}^1 \times \Delta^*_\epsilon \tag{5.21}$$

for some (multiple-valued) analytic functions $P(w), Q(w)$ in $\Delta^*_\epsilon$, where $P(w)$ is not a zero function, as (5.21) contains the graph $\{z = \psi(w)\} = Q_p$. Hence, there exists a (multiple-valued) analytic mapping $h_p(w) : \Delta^*_\epsilon \longrightarrow \mathbb{CP}^1$ such that the graph $\{z = h_p(w)\}$ is contained in (5.21) (in fact, the union of the graph with a countable collection of horizontal projective lines $\{w = const\}$ is given by (5.21). The latter means that $Q_p$ is contained in the graph $z = h_p(w)$, as required for statement (ii).

To prove (iii) we note that the mapping $\mathcal{F}$ is single-valued if and only if the functions $\alpha_j(w)$, $\beta_j(w)$, $j = 0, 1, 2$, can be scaled to be single-valued. Now for each Segre variety $Q_p$ of a point in $p = (p_1, p_2)$,

$p_1, p_2 \neq 0$, we may represent $h_p(w)$ explicitly, using (5.21), as

$$h_p(w) = -\frac{\lambda_0\beta_0(w) + \lambda_1\beta_1(w) + \lambda_2\beta_2(w)}{\lambda_0\alpha_0(w) + \lambda_1\alpha_1(w) + \lambda_2\alpha_2(w)}. \tag{5.22}$$

Since $\mathcal{F}$ is locally biholomorphic, the parameter $\lambda \in \mathbb{CP}^2$ in (5.22) runs over some open subset of $\mathbb{CP}^2$. This implies that $h_p(w)$ as in (5.22) is single-valued for all $p \in U \setminus X$ if and only if the functions $\alpha_j(w)$ and $\beta_j(w)$ can be scaled to be single-valued. This completes the proof of (iii).

The proof of (iv) also uses representation (5.22) and is analogous to (iii). However, one needs to take care of certain details. Suppose first that $\mathcal{F}$ extends holomorphically to $X$. Replacing $\mathcal{F}$ by $\sigma \circ \mathcal{F}$ for some $\sigma \in \mathrm{Aut}(\mathbb{CP}^2)$ if necessary, we may use the representation (5.15). For each $z \in \Delta_r$ consider the discrete set $E_z = \{w \in \Delta_\epsilon^* : z = -\delta(w)\}$. Then, considering the two expressions $f|_{z=z_0}, g|_{z=z_0}$ as in (5.15) for a fixed $z_0 \in \Delta_r$, we conclude that these expressions, defined on the set $E_{z_0}$, extend to $w = 0$ meromorphically. Hence, they extend meromorphically to the disc $\Delta_\epsilon$. The latter fact, applied to an arbitrary $z_0 \in \Delta_r$, implies that $\alpha(w), a(w), \beta(w), b(w), \delta(w) \in \mathcal{M}(0)$. We conclude that the functions $\alpha_j(w), \beta_j(w) \in \mathcal{M}(0)$ in (5.22), so that $h_p(w) \in \mathcal{M}(0)$, as required.

Suppose now that each of the functions $h_p(w) \in \mathcal{M}(0)$. After taking a composition of $\mathcal{F}$ with an element of $\mathrm{Aut}(\mathbb{CP}^2)$, the representation (5.15) can be applied (note that, from statement (iii) of the theorem, all functions in (5.15) are single-valued). Then (5.22) takes the form

$$h_p(w) = -\delta(w) - \frac{\lambda_1\alpha(w) + \lambda_2 a(w)}{\lambda_0 + \lambda_1\beta(w) + \lambda_2\beta(w)}. \tag{5.23}$$

Using the fact that the right-hand side in (5.23) belongs to the class $\mathcal{M}(0)$ for arbitrary $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{CP}^2$, we conclude that $\alpha(w), a(w), \beta(w), b(w), \delta(w) \in \mathcal{M}(0)$. We then can assume, performing in (3.6) scaling by an appropriate $w^l, l \in \mathbb{Z}$, that the functions $\alpha_j(w), \beta_j(w) \in \mathcal{O}(0)$ in (3.6) and, moreover, that at least one of the six functions is nonzero at $w = 0$. It is then clear that (3.6) with $\alpha_j(w), \beta_j(w) \in \mathcal{O}(0)$ allows us to extend the mapping $\mathcal{F}$ to any point $(z_0, 0) \in X$, $z_0 \in \Delta_r$, unless there exists $z_0 \in \Delta_r$ such that

$$\alpha_0(0)z_0 + \beta_0(0) = \alpha_1(0)z_0 + \beta_1(0) = \alpha_2(0)z_0 + \beta_2(0) = 0.$$

We claim that this is not possible. Indeed, assume, without loss of generality, that $z_0 = 0$. Then $\beta_0(0) = \beta_1(0) = \beta_2(0) = 0$, and for some $j \in \{0, 1, 2\}$ we have $\alpha_j(0) \neq 0$. Applying now (5.22), we conclude that for an appropriate open dense set of the projective line, determined by an element $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{CP}^2$, the corresponding function $z = h(w)$, as in (5.22), satisfies $h(0) = 0$. Denote by $\mathcal{Q} \subset \mathbb{CP}^2$ the quadric, containing $\mathcal{F}(M \setminus X)$. Since the set of projective lines $L$ in $\mathbb{CP}^2$ with $L \cap \mathcal{Q} = \emptyset$ is open, we choose a graph $z = h(w)$, as in (5.22), such that $h(0) = 0$ and $\mathcal{F}(\{z = h(w), w \neq 0\}) \cap \mathcal{Q} = \emptyset$. However, $\mathcal{F}(\{z = h(w), w \neq 0\}) \cap \mathcal{Q}$ contains the set

$$\mathcal{F}(\{z = h(w),\ \mathrm{Im}\, w = \rho(h(w), \overline{h(w)}, \mathrm{Re}\, w),\ 0 < |w| < \epsilon\}),$$

where $\mathrm{Im}\, w = \rho(z, \bar{z}, \mathrm{Re}\, w)$ is the defining function of the hypersurface $M$ with $d\rho(0) = 0$. Since $\{z = h(w),\ \mathrm{Im}\, w = \rho(h(w), \overline{h(w)}, \mathrm{Re}\, w),\ |w| < \epsilon\} \subset M$ is a nonconstant real curve passing through the origin and $\mathcal{F}$ is locally biholomorphic for $w \neq 0$, we obtain a contradiction. The proof for $0 < |z_0| < r$ is analogous. q.e.d.

**5.4. Proof of statements (ii) and (iii) of Theorem 3.3.** The following two computations furnish the proof of part (ii).

**Proposition 5.3.** *The following relations hold for the equation* (3.2)*:*

$$C(w) = -\frac{1}{9}A^2(w), \quad D(w) = \frac{1}{3}w^{2m}\left(\frac{A(w)}{w^m}\right)' - \frac{1}{3}A(w)B(w). \tag{5.24}$$

*Proof.* By taking the composition with an appropriate element $\sigma \in \mathrm{Aut}(\mathbb{CP}^2)$, we choose the associated mapping $\mathcal{F}$ to be given as in (5.15) with $\delta \neq 0$. Using the representations $I_2 = -\frac{Az+B}{w^m}$ and $I_3 = -\frac{Cz^3+Dz^2+Ez+F}{w^{2m}}$ from (3.2) and (3.1), and applying (5.18) and (5.19), we obtain

$$-\frac{A(w)}{3w^m} = \frac{b'\alpha' - \beta' a'}{a'\alpha - \alpha' a}, \quad -\frac{C(w)}{w^{2m}} = \frac{\beta' b'' - b'\beta''}{a'\alpha - \alpha' a}.$$

We let $k(w) = \frac{a(w)}{\alpha(w)}$. Using (5.17) we conclude that $k(w)$ is not a constant. Then, using (5.16) and expressing everything in terms of $k$, $\alpha$, and $\beta$, we calculate that

$$b' = k\beta', \quad \frac{A(w)}{3w^m} = \frac{\beta'}{\alpha}, \quad \frac{C(w)}{w^{2m}} = -\frac{\beta'^2}{\alpha^2}, \tag{5.25}$$

and so $C(w) = -\frac{1}{9}A^2(w)$. Further, (5.18) and (5.19) show that

$$-\frac{B(w)}{w^m} = \frac{a\alpha'' - \alpha a''}{a'\alpha - \alpha' a} - \frac{A(w)}{w^m}\delta,$$

$$-\frac{D(w)}{w^{2m}} = \frac{\beta' a'' - b'\alpha'' + \alpha' b'' - a'\beta''}{a'\alpha - \alpha' a} - \frac{3C(w)}{w^{2m}}\delta.$$

Expressing everything in terms of $k$, $\alpha$, $\beta$, and $\delta$ again gives (5.24).

q.e.d.

**Proposition 5.4.** *The following relations hold between the ODE* (3.2) *and the exponential defining equation* (3.1) *of an $m$-nonminimal* *hypersurface $M \in \mathcal{P}_0$:*

$$F(w) = 2\varphi_{23}(w),\ A(w) = \pm 6i\varphi_{32}(w),\ B(w) = \pm 2i\varphi_{22}(w) - w^{m-1},$$

$$E(w) = 6\varphi_{33} \pm 2i(m-1)\varphi_{22}w^{m-1} - 8(\varphi_{22})^2 \mp 2i\varphi_{22}' w^m. \tag{5.26}$$

$$A(w) = \pm 3i\bar{F}(w).$$

*Proof.* Consider the case when $M$ is positive. We use the form $Q_{(a,b)} = \{w = \bar{b}e^{i\varphi(z,\bar{a},\bar{b})}\}$ for Segre varieties of $M$ and substitute this representation into (3.2). As a result we obtain an identity for two power series in $z, \bar{a}, \bar{b}$. We rewrite both sides of this identity as power series in $z$ and $\bar{a}$ with coefficients depending on $\bar{b}$. If we equate the coefficients of $\bar{a}^3$, we obtain $2\phi_{23}(\bar{b}) = F(\bar{b})$. If we equate the terms $z\bar{a}^2$ we obtain $6i\phi_{32}(\bar{b}) = A(\bar{b})$. Similar computations for $\bar{a}^2$ and $z^3\bar{a}$ give the formulas for $B$ and $E$.

In order to prove the relation $A(w) = 3i\bar{F}(w)$ we consider the reality condition (2.2) as equality of power series in $z$, $\bar{z}$, and $\bar{w}$, and compare the terms with $z^3\bar{z}^2$. Taking into account that $\varphi$ does not contain $z^2\bar{z}$-degree terms (as $M \in \mathcal{P}_0$), we get $\varphi_{32}(w) = \bar{\varphi}_{23}(w)$, which gives, using (5.26), $A(w) = 3i\bar{F}(w)$, as required. The proof in the negative case is analogous. q.e.d.

Propositions 5.3 and 5.4 prove statement (ii) of Theorem 3.3.

To prove statement (iii) we argue as in the proof of statement (ii) of Theorem 3.4 and conclude that there exists a possibly smaller associated neighbourhood $U$ such that each Segre variety $Q_p$, $p \in \Delta_\delta^* \times \Delta_\epsilon^*$, is the graph of an injective function $w_p(z)$, so that it can be also represented as a graph $z = z_p(w)$. It is straightforward then to recalculate the derivatives:

$$w_z = \frac{1}{z_w},\ w_{zz} = \left(\frac{1}{z_w}\right)_w \cdot w_z = -\frac{z_{ww}}{(z_w)^3}.$$

Substituting these into (3.2) we obtain (3.5), so that all the functions $z_p(w)$ satisfy (3.5).

The injectivity of the correspondence $M \longrightarrow \mathcal{E}(M)$ follows from statement (ii). This completely proves the theorem.

## 6. Associated equation and the analytic continuation

The main conclusion that can be drawn from the results of the previous section is that we can associate with a hypersurface $M \subset \mathbb{C}^2$ of class $\mathcal{P}_0$ the complex differential equation $\mathcal{E}(M)$, given by (3.5) and satisfying the relations (3.4), in such a way that the Segre varieties of $M$ are open domains on the graphs of solutions of the equation $\mathcal{E}(M)$. In particular, statements (iii) and (iv) of Theorem 3.4 admit the following ODE-interpretation:

*All solutions in the annulus $\Delta^*_\epsilon$ of the equation $\mathcal{E}(M)$ exist as globally defined, possibly multiple-valued, analytic mappings $h : \Delta^*_\epsilon \longrightarrow \mathbb{CP}^1$. Furthermore:*

*(iii)′ The analytic mapping $\mathcal{F} : U \setminus \{w = 0\} \longrightarrow \mathbb{CP}^2$ associated with $M$ is single-valued if and only if all solutions of the equation $\mathcal{E}(M)$ are single-valued mappings $\Delta^*_\epsilon \longrightarrow \mathbb{CP}^1$.*

*(iv)′ The analytic mapping $\mathcal{F} : U \setminus \{w = 0\} \longrightarrow \mathbb{CP}^2$ associated with $M$ extends to the complex locus $\{w = 0\}$ holomorphically if and only if all local solutions of the equation $\mathcal{E}(M)$ extend meromorphically to $\Delta_\epsilon$.*

Statements (iii)’ and (iv)’ now give a hint on how to prove Theorem 3: we need to show the moderate growth of solutions of the ODE $\mathcal{E}(M)$ as $w \longrightarrow 0$. This allows us to reduce Theorem 3 to a question that can be formulated purely in terms of analytic theory of differential equations. Realization of this strategy is the content of Sections 6 and 7.

**6.1. Fuchsian and non-Fuchsian hypersurfaces.** Equation $\mathcal{E}(M)$ obtained in Section 5 is an ordinary second order meromorphic differential equation defined in the domain $\Delta_\epsilon \subset \mathbb{C}$. $\mathcal{E}(M)$ is polynomial w.r.t. the unknown function $z$ and its derivative $z'$, and has in $\Delta_\epsilon$ a unique (and hence isolated) meromorphic singularity at the point $w = 0$. The study of this type of equations was initiated by Poincaré and Painlevé (see [**43**], [**19**], [**3**], [**54**]), and it continues to be an active area of research (see, for example, [**22**], [**9**], [**35**], [**20**], [**25**] and references therein). In his celebrated work [**40**] Painlevé classified second order complex ODEs, rational in the dependent variable $z$ and its derivative, meromorphic in some domain $\Omega$ in the independent variable $w$, and having no movable critical points (ODEs of this type are called *ODEs of class $\mathcal{P}$).* The mapping, bringing an ODE of class $\mathcal{P}$ to its standard form in this classification, is locally biholomorphic in $\mathbb{CP}^1 \times \Omega$ and is linear-fractional in the dependent variable (see, e.g., [**3**]). Note that the associated mapping $\mathcal{F}$, considered in the present paper, has the above described form and brings the associated ODE $\mathcal{E}(M)$ to its standard form $z'' = 0$. Thus real hypersurfaces, considered in the paper, are associated with ODEs of class $\mathcal{P}$ with the simplest standard form $z'' = 0$. This explains the $\mathcal{P}_0$-notation for them.

As explained in Section 2, in the particularly important *linear* case the behaviour of solutions for the ODE $\mathcal{E}(M)$ is characterized by the Fuchsian condition. The Fuchsian type condition for a hypersurface $M \in \mathcal{P}_0$, described in Introduction, can be stated in terms of the associated equation $\mathcal{E}(M)$ and is imposed by a similarity with the linear case. To show that, we first observe that *a hypersurface $M \in \mathcal{P}_0$ satisfies the Fuchsian type condition if and only if the associated equation $\mathcal{E}(M)$*

*satisfies*

$$ord_0 B(w) \geq m-1,\ ord_0 E(w) \geq 2m-2, \tag{6.1}$$
$$ord_0 A(w) = ord_0 F(w) \geq \frac{3}{2}(m-1).$$

The formulated statement follows directly from formulas (3.3). Here for a nonzero function $h(w) \in \mathcal{M}(0)$ we denote by $\mathrm{ord}_0 h$ the order of vanishing of $h$ if it is holomorphic at 0, and the negative order of pole for $h$ otherwise.

Next we investigate the Fuchsian type condition. For that we introduce an alternative to (6.1) description.

**Definition 6.1.** A hypersurface $M \in \mathcal{P}_0$ is called *$l$-reducible, $l \in \mathbb{Z}$, if the change of variables $Z = zw^l$, $W = w$ brings the associated ODE $\mathcal{E}(M)$ to an ODE of the form*

$$Z'' = \frac{1}{W}(\hat{A}Z + \hat{B})Z' + \frac{1}{W^2}(\hat{C}Z^3 + \hat{D}Z^2 + \hat{E}Z + \hat{F}) \tag{6.2}$$

*for some holomorphic near the origin functions $\hat{A}(W), \hat{B}(W), \hat{C}(W), \hat{D}(W), \hat{E}(W), \hat{F}(W)$.*

The $l$-reducibility condition turns out to be equivalent to the Fuchsian type. In particular, it is a biholomorphic invariant of $M$.

**Proposition 6.2.**

*(1) A hypersurface $M \in \mathcal{P}_0$ is of Fuchsian type if and only if the associated ODE $\mathcal{E}(M)$ is $l$-reducible for some $l \geq 0$. Moreover, $l$ can be chosen in such a way that the polynomial $\hat{C}(0)t^3 + \hat{D}(0)t^2 + \hat{E}(0)t + \hat{F}(0)$ is not a nonzero constant.*

*(2) The Fuchsian type condition for a nonminimal hypersurface $M \subset \mathbb{C}^2$, spherical in the complement to the complex locus, is biholomorphically invariant. In particular, this condition is independent of the choice of prenormal coordinates.*

*Proof.* (1) Suppose first that $F(w) \equiv 0$ in $\mathcal{E}(M)$. It follows from (3.4) that $A = C = D = F \equiv 0$, and the equation $\mathcal{E}(M)$ is linear. In this case it can be seen immediately that the Fuchsian type condition is equivalent to $\mathcal{E}(M)$ being Fuchsian in the sense of theory of linear ODEs, which means 0-reducibility. Moreover, the polynomial $\hat{C}(0)t^3 + \hat{D}(0)t^2 + \hat{E}(0)t + \hat{F}(0)$ has a root $t_0 = 0$, which proves the proposition under the assumption $F(w) \equiv 0$.

Consider now the case when $F \not\equiv 0$. Suppose first that $M$ is $l$-reducible for some $l \in \mathbb{Z}$. Perform in the equation $\mathcal{E}(M)$ associated with the hypersurface $M \in \mathcal{P}_0$ the change of variables $Z = zw^l$, $W = w$, and rewrite the new equation in the form $Z'' = (p_1 Z + p_0)Z' + (q_3 Z^3 + q_2 Z^2 + q_1 Z + q_0)$ for certain $p_i, q_j \in \mathcal{M}(0)$. Then, by recalculating the derivatives and substituting them into $\mathcal{E}(M)$, it is not difficult to check that the properties $\mathrm{ord}_0 p_0 \geq -1$ and $\mathrm{ord}_0 q_1 \geq -2$ hold simultaneously if and only if the terms $\frac{B(w)}{w^m}$ and $\frac{E(w)}{w^{2m}}$ have the same properties simultaneously, so that from $l$-reducibility we have $\mathrm{ord}_0 B \geq m-1$, $\mathrm{ord}_0 E \geq 2m-2$. Also we compute that $\mathrm{ord}_0 q_0 = \mathrm{ord}_0 F + l - 2m$. From the $l$-reducibility, $\mathrm{ord}_0 q_0 = -2+s$ for some integer $s \geq 0$, and thus $l = 2m-2+s-\mathrm{ord}_0 F$. From (5.26) we have $\mathrm{ord}_0 A = \mathrm{ord}_0 F$, $\mathrm{ord}_0 C = 2\mathrm{ord}_0 F$, so that, after a computation, $\mathrm{ord}_0 p_1 = 2\mathrm{ord}_0 F - 3m - s + 2$. From the $l$-reducibility now $2\mathrm{ord}_0 F - 3m - s + 2 \geq -1$, and we obtain $2\mathrm{ord}_0 F \geq s + 3(m-1) \geq 3(m-1)$, as required for the Fuchsian type.

Suppose now that $M$ is of Fuchsian type. Put $l := \mathrm{ord}_0 F - m + 1$. Now arguing as above and using $\mathrm{ord}_0 \frac{B(w)}{w^m} \geq -1$, $\mathrm{ord}_0 \frac{E(w)}{w^{2m}} \geq -2$, $\mathrm{ord}_0 A = \mathrm{ord}_0 F$, $\mathrm{ord}_0 C = 2\mathrm{ord}_0 F$, we get $\mathrm{ord}_0 p_0 \geq -1$, $\mathrm{ord}_0 q_1 \geq -2$, $\mathrm{ord}_0 q_0 = \mathrm{ord}_0 F + l - 2m \geq -2$, $\mathrm{ord}_0 p_1 = \mathrm{ord}_0 A - l - m = -1$, $\mathrm{ord}_0 q_3 = \mathrm{ord}_0 C - 2l - 2m = -2$, $\mathrm{ord}_0 q_2 \geq -2$, so that we obtain an equation of the form required for $l$-reducibility. The integer $l$ here is equal to $\mathrm{ord}_0 F - m + 1 \geq \frac{m-1}{2}$ and thus is nonnegative. To check that the polynomial $\hat{C}(0)t^3 + \hat{D}(0)t^2 + \hat{E}(0)t + \hat{F}(0)$ is not a constant, we note that for the latter choice of $l$ we have $\mathrm{ord}_0 q_3 = -2$, so that $\hat{C}(0) \neq 0$. This finally proves (1).

In order to prove (2) we consider two hypersurfaces $M, \tilde{M} \in \mathcal{P}_0$ and a local biholomorphism $G : (M,0) \longrightarrow (\tilde{M},0)$ between them. Suppose that $M$ is of Fuchsian type. Then, according to (1), the transformation $H : (z,w) \longrightarrow (zw^l, w)$ for an appropriate integer $l \geq 0$ brings $\mathcal{E}(M)$ into an ODE of the form (6.2). Hence, the transformation $H \circ G^{-1}$, which has the form $(f(z,w)w^l + O(|z|^2|w|^l) + O(|w|^{l+1}), g(z,w))$ for an appropriate local biholomorphism $(f,g) : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$, brings $\mathcal{E}(\tilde{M})$ into an ODE of the form (6.2). Arguing now similarly to the proof of (1) we deduce from here that $\tilde{M}$ is of Fuchsian type, which proves statement (2) and the proposition. q.e.d.

**Definition 6.3.** Let $M \in \mathcal{P}_0$ be an $m$-nonminimal hypersurface of Fuchsian type. The ODE

$$Z'' = \frac{1}{W}(\hat{A}Z + \hat{B})Z' + \frac{1}{W^2}(\hat{C}Z^3 + \hat{D}Z^2 + \hat{E}Z + \hat{F}),$$

obtained from $\mathcal{E}(M)$ by the change of variables $Z = zw^l$, $W = w$ with $l := \mathrm{ord}_0 F - m + 1 \geq 0$ (as in the proof of Proposition 6.2), is called *the associated ODE $\mathcal{E}^r(M)$.*

According to Proposition 6.2, the associated ODE $\mathcal{E}^r(M)$ always exists in the Fuchsian type case, and the polynomial $\hat{C}(0)t^3 + \hat{D}(0)t^2 + \hat{E}(0)t + \hat{F}(0)$ is not a nonzero constant.

**6.2. Hypersurfaces with rotational symmetries. Examples.** The associated ODE $\mathcal{E}(M)$ is particularly simple in the special case when a hypersurface $M \in \mathcal{P}_0$ is invariant under the group $(z,w) \longrightarrow (e^{it}z,\, w)$, $t \in \mathbb{R}$, of rotational symmetries. As each above rotational symmetry sends a Segre variety of $M$ into another Segre variety, it must be a symmetry of the ODE $\mathcal{E}(M)$, and it is not difficult to see that the associated ODE $\mathcal{E}(M)$ is linear in the rotational case. Thus we conclude that *Theorems 2 and 3.5 follow from the Fuchs theorem in the rotational case.* This also shows that the regularity condition in Theorem 3 (namely, the Fuchsian type condition) is optimal in the rotational case.

**Remark 6.4.** As follows from the described connection between rotational hypersurfaces of class $\mathcal{P}_0$, Theorem 3.15 in [**33**] and Theorem 3.3 of the present paper, *the algorithm for obtaining nonminimal spherical hypersurfaces with rotational symmetries, described in Remark 3.18 in* [**33**]*, gives a complete description of hypersurfaces of class $\mathcal{P}_0$ with rotational symmetries.*

However, as the example of hypersurfaces $M_{R,0}$ in [**31**] shows, the investigation of nonminimal spherical hypersurfaces in $\mathbb{C}^2$ *cannot* be reduced to the rotational case. Below we demonstrate applications of Theorems 1 and 2 (or, alternatively, Theorems 3.4 and 3.6) and give explicit examples of the associated ODE construction in the rotational case.

**Example 6.5.** The 1-nonminimal hypersurfaces $L_s$, $s \in \mathbb{R}, s \neq 0$, with the complex locus $\{w = 0\}$, given by

$$v = u \tan\left(\frac{1}{s}\ln(1 + s|z|^2)\right),$$

were obtained in [**8**] as examples of nonminimal hypersurfaces with 4-dimensional infinitesimal automorphism algebras (see also [**31**]). It is not difficult to check that each $L_s$ is of class $\mathcal{P}_0$. Indeed, one has to check only the sphericity of $L_s$ at Levi nondegenerate points, and this follows from the fact that only spherical hypersurfaces admit $\geq 4$ dimensional infinitesimal automorphism algebras at Levi nondegenerate points [**6**]. The complex defining equation of $L_s$ has the form $w = \bar{w}\exp\left(\frac{2i}{s}\ln(1 + sz\bar{z})\right)$. For a point $(a,b) \in \mathbb{C}^2$ with $a, b \neq 0$ its Segre variety $Q_{(a,b)}$ equals (locally)

$$z(w) = h_{(a,b)}(w) = \frac{1}{s\bar{a}}\left(\frac{w}{\bar{b}}\right)^{\frac{s}{2i}} - \frac{1}{s\bar{a}}.$$

Clearly, for any $s \in \mathbb{R}, a, b \in \mathbb{C}, s, a, b \neq 0$, the germ $h_{(a,b)}(w)$ does not extend to the origin meromorphically, so by Theorem 2 the associated mapping $\mathcal{F}$ *does not* extend to the complex locus holomorphically.

The next example illustrates in detail the connection between a family of hypersurfaces $M_\gamma \in \mathcal{P}_0$, the associated ODEs $\mathcal{E}(M_\gamma)$, and the associated mappings $\mathcal{F}_\gamma$.

**Example 6.6.** For the 1-nonminimal hypersurfaces $M_\gamma \subset \mathbb{C}^2$, $\gamma \in \mathbb{R} \setminus \{0\}$, containing the complex hypersurface $X = \{w = 0\}$ and given in a neighbourhood of the origin by

$$w = \bar{w}\left(i|z|^2 + \sqrt{1 - |z|^4}\right)^{\frac{1}{\gamma}}$$

(see [**31**]), the family of Segre varieties near the origin has the form $Q_{(a,b)} = \{w = \bar{b}(iz\bar{a} + \sqrt{1 - z^2\bar{a}^2})^{\frac{1}{\gamma}}\}$. Elementary computations show that $Q_{(a,b)}$ with $a, b \neq 0$ are open domains on the graphs

$$\tilde{Q}_{(a,b)} = \left\{ z = \frac{1}{2i\bar{a}}\left(\frac{w^\gamma}{\bar{b}^\gamma} - \frac{\bar{b}^\gamma}{w^\gamma}\right)\right\}. \tag{6.3}$$

By Theorem 2, the associated mapping $\mathcal{F}_\gamma$ extends to the complex locus holomorphically if and only if $\gamma \in \mathbb{Z}$. In fact one can see that $F_\gamma$ is given by $z \longrightarrow zw^\gamma$, $w \longrightarrow w^{2\gamma}$.

Following the elimination process described in Section 2 it is not difficult to conclude that all the graphs $\tilde{Q}_{(a,b)}$, $a, b \neq 0$ satisfy the linear ODE

$$z'' = -\frac{1}{w}z' + \frac{\gamma^2}{w^2}z,$$

which coincides, by uniqueness, with $\mathcal{E}(M_\gamma)$. This ODE is Fuchsian for any $\gamma \in \mathbb{R}$.

The next two examples show that for $m > 1$ the ODE $\mathcal{E}(M)$ associated with a hypersurface of class $\mathcal{P}_0$ may be both of Fuchsian and non-Fuchsian type.

**Example 6.7.** Consider the $m$-nonminimal with $m \geq 2$ hypersurfaces $M_0^m \in \mathcal{P}_0$ (see [**33**]), given near the origin by the complex defining equations

$$w = \bar{w}\left(1 + \frac{i}{2}(1 - m)\bar{w}^{m-1}\ln\frac{1}{1 - 2|z|^2}\right)^{\frac{1}{1-m}}. \tag{6.4}$$

The Levi nondegenerate part of $M_0^m$ is the preimage of a domain in the quadric $\mathcal{Q} = \left\{2|Z|^2 + |W|^2 = 1\right\} \subset \mathbb{C}^2$ under the single-valued mapping

$$\Lambda_m : (Z, W) = \left(z,\, e^{\frac{2i}{1-m}w^{1-m}}\right).$$

It follows that the mapping $\Lambda_m$ is associated with $M_0^m$. Remarkably, *each mapping* $\Lambda_m$ *does not extend to the complex locus* $\{w = 0\}$, *even though it is single-valued.* From the elimination procedure from Section 2 (or the arguments from [**33**]), we conclude that the associated ODE $\mathcal{E}(M_0^m)$ is of non-Fuchsian form $z'' = \left(\frac{2i}{w^m} - \frac{m}{w}\right)z'$. This agrees with Theorem 3.

**Example 6.8.** For the 2-nonminimal hypersurface $M \in \mathcal{P}_0$, given by $v = (u^2 + v^2)|z|^2$, it is not difficult to see that the polynomial mapping $\mathcal{F}(z, w) = (zw, w)$ maps $M$ into the hyperquadric $\{\mathrm{Im}\, w = |z|^2\} \subset \mathbb{C}^2$. The associated ODE $z'' = -\frac{2}{w} z'$ is Fuchsian.

**Remark 6.9.** As the family of hypersurfaces $M^m_\beta \in \mathcal{P}_0$ in [**33**] shows, the associated mapping $\mathcal{F}$ cannot be in general expressed in terms of elementary functions when $m > 1$, even though the associated ODE is given by elementary functions. In this case the extension/no extension dichotomy can be resolved *only* using the associated equation $\mathcal{E}(M)$ and Theorem 3.

**6.3. Reduction of Theorem 3 to the existence of a holomorphic solution.** In this subsection we perform an important step toward the proof of sufficiency in Theorem 3, reducing it to Theorem 3.5, i.e., the question that can be formulated purely in terms of analytic theory of differential equations.

**Proposition 6.10.** *Suppose that an $m$-nonminimal hypersurface $M \in \mathcal{P}_0$ is of Fuchsian type and the associated mapping $\mathcal{F}$ is single-valued. Suppose, in addition, that the associated equation $\mathcal{E}^r(M)$ admits a holomorphic at the origin solution $z = h(w)$. Then $\mathcal{F}$ extends to the complex locus $X = \{w = 0\}$ holomorphically.*

*Proof.* We choose $l := \mathrm{ord}_0 F - m + 1 \geq 0$ as in the definition of the ODE $\mathcal{E}^r(M)$, and reduce the ODE $\mathcal{E}(M)$ to the ODE $\mathcal{E}^r(M)$ by the change of variables $Z = zw^l$, $W = w$. Using Theorem 3.4, we represent all solutions of the equation $\mathcal{E}(M)$ in the form (5.22) with single-valued $\alpha_0(w), ..., \beta_2(w)$. We introduce a locally biholomorphic mapping $\widehat{\mathcal{F}} : \mathbb{C}^1 \times \Delta^*_\epsilon \longrightarrow \mathbb{CP}^2$ given by

$$(Z, W) \longrightarrow (\hat{\alpha}_0(W)Z + \hat{\beta}_0(W), \hat{\alpha}_1(W)Z + \hat{\beta}_1(W), \hat{\alpha}_2(W)Z + \hat{\beta}_2(W)),$$

where the single-valued functions $\hat{\alpha}_j, \hat{\beta}_j$ are defined as $\hat{\alpha}_j := \frac{1}{w^l}\alpha_j$, $\hat{\beta}_j := \frac{1}{w^l}\beta_j$. According to Theorem 3.4, it is sufficient to prove that the collection of functions $\alpha_j, \beta_j$ can be scaled to belong to the class $M(0)$. Obviously, it is sufficient to prove the same fact for the collection $\hat{\alpha}_j, \hat{\beta}_j$.

Since $Z = h(W)$ is a solution of the ODE $\mathcal{E}^r(M)$, the mapping $\widehat{\mathcal{F}}$ sends $\{Z = h(W)\}$ into some projective hyperplane in $\mathbb{CP}^2$. We then compose $\widehat{\mathcal{F}}$ with an element of $\mathrm{Aut}(\mathbb{CP}^2)$ in such a way that $\{Z = h(W)\}$ is mapped into $\mathbb{CP}^2 \setminus \mathbb{C}^2$. Using the representation of type (5.15) for the mapping $\widehat{\mathcal{F}}$ with appropriate functions $\hat{\alpha}(W), \hat{a}(W), \hat{\beta}(W), \hat{b}(W), \hat{\delta}(W)$, we conclude that $\hat{\delta}(W) = -h(W) \in \mathcal{O}(0)$. Arguments similar to those in the proof of Theorem 3.3 show that the fact that $\widehat{\mathcal{F}}$ transforms the ODE $\mathcal{E}$ into $(Z^*)'' = 0$ yields formulas identical to (5.18),(5.19) in terms

of $\hat\alpha(W),\hat a(W),\hat\beta(W),\hat b(W),\hat\delta(W)$. Set $\hat k(W):=\frac{\hat a(W)}{\hat\alpha(W)}$. Then
(6.5)

$$\hat b'=\hat k\hat\beta',\ \frac{\hat\beta'}{\hat\alpha}=\frac{\hat A(W)}{3w},\ \hat a'\hat\alpha-\hat\alpha'\hat a=\hat k'\hat\alpha^2,\ \frac{(\hat k'\hat\alpha^2)'}{\hat k'\hat\alpha^2}=\frac{\hat A(W)}{W}\hat\delta-\frac{\hat B(W)}{W}.$$

Formulas (6.5) show that if $\hat\alpha\in M(0)$, then $\hat\beta,\hat k,\hat a,\hat b\in M(0)$. The reason is that if a meromorphic in a punctured disc $\Delta^*_\epsilon(0)$ function $u(W)$ satisfies $\frac{Wu'}{u}\in\mathcal{O}(0)$, then $u\in M(0)$.

To verify the fact $\hat\alpha\in M(0)$, we continue a detailed expansion of (5.19), using (6.5), in terms of $\hat\alpha,\hat k$. Then a computation shows that

$$-\frac{\hat E(W)}{W^2}=\left(\frac{\hat B(W)}{W}-\frac{\hat A(W)}{W}\hat\delta\right)\frac{\hat\alpha'}{\hat\alpha}+\frac{\hat\alpha''}{\hat\alpha}-\hat\delta'\frac{\hat A(W)}{W}-2\delta\frac{\hat D(W)}{W^2}-3\delta\frac{\hat C(W)}{W^2}.$$

The obtained equality can be considered as a second order Fuchsian ODE with the unknown function $\hat\alpha(W)$. By the Fuchs theorem we conclude that $\hat\alpha(W)\in M(0)$, which proves $\hat\alpha,\hat\beta,\hat a,\hat b,\hat\delta\in M(0)$. Hence, the collection $\hat\alpha_j,\hat\beta_j$ can be scaled to become holomorphic at $W=0$, as required. q.e.d.

## 7. Existence of a holomorphic solution

By the results of the previous section, in order to prove Theorem 2 we need to show that equation $\mathcal{E}^r(M)$ associated with an $m$-nonminimal Fuchsian type hypersurface $M\in\mathcal{P}$ admits a holomorphic at the origin solution $z=h(w)$, provided its solutions are single-valued. In this section we prove a more general fact (Theorem 3.5), stating that *any* ODE similar to $\mathcal{E}^r(M)$ must have at least one holomorphic at the origin solution, provided that no solution can branch about the origin.

Section 6.1 shows that if the ODE $\mathcal{E}^r(M)$ associated with a Fuchsian type hypersurface $M\in\mathcal{P}$ is such that the associated mapping $\mathcal{F}$ is single-valued, then it satisfies the conditions of Theorem 3.5. To see this it is enough to choose $z_0$ as a root of the polynomial $\hat C(0)t^3+\hat D(0)t^2+\hat E(0)t+\hat F(0))$. Hence, Theorem 3.5 implies Theorems 3.6 and 3.

The idea of the proof of Theorem 3.5 is as follows: The result is trivial if the function $Q(z,w)$ is independent of $w$ because we may simply take $z(w):=z_0$ as a holomorphic solution. For the general case we apply the Poincaré Small Parameter Method. Further, thanks to the convergence result in [**20**] (see Theorem A.12 there), in order to prove Theorem 3.5 it is sufficient to prove the existence of a *formal* holomorphic solution for the equation $\mathcal{E}$, as any such solution is automatically convergent, without any assumption on the eigenvalues of the linearization matrix. We note that the convergence result can be also proved using the standard technique of majorizing functions, but we do not provide the proof here. By a *formal holomorphic solution* for the equation $\mathcal{E}$ we mean a formal power series $z(w) = \sum_{r=0}^{\infty} a_r w^r$, that makes $\mathcal{E}$ an identity of two Laurent series in $w$ (with finite principal parts).

After a simple substitution $z \longrightarrow z - z_0$ we may assume $z_0 = 0$. Thus, for the proof of Theorem 3.5 it remains to prove the following

**Theorem 7.1.** *In the assumptions of Theorem 3.5 with $z_0 = 0$, the equation $\mathcal{E}$ admits a formal solution $z(w) = \sum_{r=1}^{\infty} a_r w^r$.*

*Proof.* We represent equation $\mathcal{E}$ as a system by introducing a new unknown function

$$u(w) := wz'(w).$$

Then we have $z' = \frac{u}{w}$ and $z'' = \frac{u'}{w} - \frac{u}{w^2}$, so that $\mathcal{E}$ becomes the system

$$\begin{cases} z' = \frac{u}{w}, \\ u' = \frac{1}{w}\left[(1 + P(z,w))u + Q(z,w)\right]. \end{cases} \tag{7.1}$$

Recall that we assume $z_0 = 0$, so that $Q(0,0) = 0$. Clearly, the existence of the desired solution is equivalent to the existence of a formal holomorphic solution $z(w) = \sum_{r=1}^{\infty} a_r w^r$, $u(w) = \sum_{r=1}^{\infty} b_r w^r$ for the system (7.1). We expand the functions $1 + P(z,w)$ and $Q(z,w)$ as $1+P(z,w) = \sum_{k,j\geq 0} p_{kj} z^k w^j$, $Q(z,w) = q_{10}z + q_{01}w + \sum_{k,j>0} q_{kj} z^k w^j$. Plugging all the power series representations into (7.1) and gathering terms with $w^{r-1}$, $r \geq 1$, we obtain

$$\begin{gathered} a_1 - b_1 = 0, \\ b_1 - p_{00}b_1 - q_{10}a_1 = q_{01}, \end{gathered} \tag{7.2}$$

for $r = 1$, and

$$\begin{gathered} ra_r - b_r = 0 \\ rb_r - p_{00}b_r - q_{10}a_r = \sum_{2\leq k+j\leq r} q_{kj} \sum_{i_1+\ldots+i_k=r-j} a_{i_1}\cdot\ldots\cdot a_{i_k} + \\ + \sum_{l=1}^{r-1} b_l \sum_{1\leq k+j\leq r-l} p_{kj} \sum_{i_1+\ldots+i_k=r-j-l} a_{i_1}\cdot\ldots\cdot a_{i_k}, \end{gathered} \tag{7.3}$$

for $r > 1$. It is presumed in (7.3) that a sum of the form $\sum a_{i_1}\cdot\ldots\cdot a_{i_k}$ equals 1 for $k = 0$. It is also *important* that for a fixed $r$ on the left-hand side, the right-hand side in both (7.2) and (7.3) contains only $a_i, b_l$ with $i, l < r$.

Now let us introduce some vector and matrix notation. We denote by $h_r \in \mathbb{C}^2$ the vector with components $a_r, b_r$, and by $L$ the $2 \times 2$ matrix $\begin{pmatrix} 0 & 1 \\ q_{10} & p_{00} \end{pmatrix}$. Then, if $I$ denotes the identity matrix, the equations (7.2),(7.3) can be rewritten for all $r \geq 1$ as:

$$(rI - L)h_r = \begin{pmatrix} 0 \\ K_r \end{pmatrix}, \tag{7.4}$$

where $K_1 = q_{01}$, and for $r \geq 2$,

$$K_r(a_1, ..., a_{r-1}, b_1, ..., b_{r-1}, \{p_{kj}\}_{1 \leq k+j \leq r-1}, \{q_{kj}\}_{2 \leq k+j \leq r})$$

is a polynomial scalar expression from the right-hand side of (7.3). It is crucial that all polynomials $K_r$ have nonnegative coefficients. We now consider two cases.

**Nonresonant case.** We assume that $L$ does not have any eigenvalues $r \in \mathbb{Z}^+$. In this case each of the equations (7.4) has a unique solution $h_r$, if $h_1, ..., h_{r-1}$ are already found, and this determines the collection $\{h_r\}_{r \geq 1}$ uniquely. We then put

$$\begin{pmatrix} z^* \\ u^* \end{pmatrix}(w) := \sum_{r=1}^{\infty} h_r w^r, \tag{7.5}$$

and $(z^*(w), u^*(w))$ becomes a formal holomorphic solution of the equation the system (7.1) by construction. This proves the theorem in the nonresonant case.

**Resonant case.** This case turns out to be much more delicate and requires additional considerations. We will prove the existence of a collection $\{h_r\}_{r \geq 1}$, satisfying (7.4), which will imply the existence of a formal holomorphic solution (7.5). Our main strategy is to show that the absence of a solution for the system of equations (7.4) leads to multiple-valuedness of certain solutions of $\mathcal{E}$, which contradicts the assumption of Theorem 3.5. In order to do that, we consider the case of a general equation $\mathcal{E}$ as a perturbation of the above "constant coefficient" case $Q = Q(z)$, by introducing a small parameter $\varepsilon$. Perform in the system (7.1) the change of variables $w = \varepsilon w^*$, $z = z^*$, $0 < |\varepsilon| < 1$, $\varepsilon \in \mathbb{C}$. In the new coordinates the system becomes

$$\mathcal{S}_\varepsilon = \begin{cases} z' = \frac{u}{w}, \\ u' = \frac{1}{w}\left[(1 + P(z, \varepsilon w))u + Q(z, \varepsilon w)\right]. \end{cases} \tag{7.6}$$

Although for the change of variables we have $\varepsilon \neq 0$, we may extend (7.6) holomorphically to $\{|\varepsilon| < 1\}$. Thus we get a holomorphic in the unit disc family $\mathcal{S}_\varepsilon$ of first-order systems. Each $\mathcal{S}_\varepsilon$ is a holomorphic perturbation of the system $\mathcal{S}_0$, that has the holomorphic solution $z = 0, u = 0$. So the strategy now is to find *analytic* solutions of $\mathcal{S}_\varepsilon$

in annuli $\{r_1 < |w| < r_2\}$, $0 < r_1 < r_2 < \epsilon$ for sufficiently small $\varepsilon$ as perturbations of the constant solution for $\mathcal{E}_0$. This general approach is known as the *Small Parameter Method*. It was invented by H. Poincaré to investigate solutions of nonlinear systems considering them as perturbations of already known solutions of initial "simple" systems. In the modern language, the method simply uses the analytic dependence of solutions of a system of first-order holomorphic ODEs on the initial conditions and holomorphic parameters, see [**22**]. We give below a convenient formulation of this

**Theorem 7.2** (Poincaré, 1892, see, e.g., [**19**].)**.** *Let $F(x, y, \varepsilon)$, $x \in \mathbb{C}$, $y \in \mathbb{C}^2$, $\varepsilon \in \mathbb{C}$, be a holomorphic function in the domain $D \times G \times E$, $x_0 \in D$ is a fixed point and $\gamma(t), 0 \leq t \leq 1$, is a smooth real-analytic path with $\gamma(t) \subset D$ and $\gamma(0) = x_0$. Suppose that $0 \in E$ and the ODE system $y' = F(x, y, 0)$ has a holomorphic solution $y_0(x)$ in a neighborhood $U$ of $[\gamma(t)]$ with $y_0(x_0) = p_0$. Then for any sequence $p_r \in \mathbb{C}^2$, $r \geq 1$, such that the power series $\sum p_r\varepsilon^r$ is convergent in some disc, and any sufficiently small $\varepsilon$, the ODE system $y' = F(x, y, \varepsilon)$ has a holomorphic w.r.t. the time $t$ on $\gamma$ solution of the form*

$$y^{\varepsilon}(\gamma(t)) = \sum_{r=0}^{\infty} y_r(t)\varepsilon^r, \tag{7.7}$$

*where $y_r(t)$, $r \geq 1$, are analytic on $[0, 1]$, with $y_r(0) = p_r$, $r \geq 0$, and the series (7.7) is uniformly convergent w.r.t. $t$ and $\varepsilon$. Each of the $y_r(t)$ extends to an open neighbourhood $\tilde{U}$ of $[\gamma]$ as a (possibly multiple-valued) analytic function $y_r(x)$ such that $y^{\varepsilon}(x) = \sum_{r=0}^{\infty} y_r(x)\varepsilon^r$ is a (possibly multiple-valued) solution of $y' = F(x, y, \varepsilon)$. Moreover, each $y_r(x)$, $r \geq 1$, is a solution of some first-order inhomogeneous linear system of ODEs with homogeneous part independent of $r$.*

We proceed now with Poincaré's Small Parameter Method. We suppose, without loss of generality, $\epsilon > 1$ (where $\{0 < |w| < \epsilon\}$ is the punctured disc where $\mathcal{E}$ is defined) and let $\gamma$ be the unit circle and $w_0 = 1 \in \gamma$ be the starting point in Poincaré's theorem. We expand

$$z^{\varepsilon}(w) = \sum_{r=1}^{\infty} z_r(w)\varepsilon^r, \quad u^{\varepsilon}(w) = \sum_{r=1}^{\infty} u_r(w)\varepsilon^r.$$

We now substitute the expansions for $z^{\varepsilon}(w)$, $1 + P(z, w)$, and $Q(z, w)$ into (7.6) and collect terms with $\varepsilon^r$, $r \geq 1$. For $r = 1$ we obtain the following inhomogeneous first-order linear ODE system in $z_1, u_1$:

$$\begin{cases} z_1' = \frac{u_1}{w}, \\ u_1' = \frac{1}{w}(p_{00}u_1 + q_{10}z_1) + q_{01}, \end{cases}$$

which can be rewritten as

$$\begin{pmatrix} z_1' \\ u_1' \end{pmatrix} = \frac{1}{w} L \begin{pmatrix} z_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ K_1 \end{pmatrix}, \tag{7.8}$$

where $L, K_1$ are as in (7.4).

**Definition 7.3.** By a *logarithmic quasipolynomial* we mean a (possibly multiple-valued) analytic in $\mathbb{C} \setminus \{0\}$ function $P(w^{\lambda_1}, ..., w^{\lambda_s}, \ln w)$, where $s \in \mathbb{Z}_{\geq 0}$, $P$ is a complex polynomial in $s+1$ variables, and $\lambda_j \in \mathbb{C}$.

We need now the following

**Lemma 7.4.** *The eigenvalues of $L$ are two distinct integers.*

*Proof.* Consider (7.8) as a inhomogeneous Euler system (see [**22**]). The characteristic roots of this system are the eigenvalues of $L$. Let $\varphi(w), \psi(w)$ be two vector-functions, forming a basis of the space of solutions for the homogeneous part of (7.8). Suppose that the eigenvalues of $L$ coincide, or at least one of them is not an integer. Then at least one of the two non-zero vector-functions $\varphi(w), \psi(w)$ (say, $\varphi(w)$) contains either a factor $w^\lambda$, $\lambda \notin \mathbb{Z}$, or a factor $w^\lambda \ln w$, $\lambda \in \mathbb{C}$, and hence is not single-valued along $\gamma$. The general solution of (7.8) has the form:

$$\begin{pmatrix} z_1 \\ u_1 \end{pmatrix} = c_1 \varphi + \tilde{c}_1 \psi + \theta_1, \tag{7.9}$$

where $c_1, \tilde{c}_1$ are constants and $\theta_1$ is a vector-function with components being logarithmic quasipolynomials (the latter fact follows from the variation of constants algorithm, applied to the Euler system, see [**22**]). We may assume, without loss of generality, $\psi(1) \neq 0$ (otherwise we replace $\gamma$ with a circle $\{|w| = R\}$ with $0 < R < 1$ and $\psi(R) \neq 0$, and take $w_0 = R$ as a starting point). Choose in (7.9) any $c_1, \tilde{c}_1$ with $c_1 \neq 0$ and $\begin{pmatrix} z_1 \\ u_1 \end{pmatrix}(1) = 0$. This fixes the term $\begin{pmatrix} z_1 \\ u_1 \end{pmatrix} \varepsilon$ in the expansion (7.7) of the solution.

We continue with the iteration process and collect terms with $\varepsilon^r$, $r \geq 2$. We obtain the following series of inhomogeneous Euler systems (with the homogeneous part identical to that in (7.8) for arbitrary $r \geq 2$):

$$\begin{pmatrix} z_r' \\ u_r' \end{pmatrix} = \frac{1}{w} L \begin{pmatrix} z_r \\ u_r \end{pmatrix} + M_r, \tag{7.10}$$

where the components of the vector-function $M_r$ are logarithmic quasipolynomials, depending on $M_j$ with $j < r$ (this again follows by induction from the variation of constants algorithm). The general solution has the form

$$\begin{pmatrix} z_r \\ u_r \end{pmatrix} = c_r \varphi + \tilde{c}_r \psi + \theta_r, \tag{7.11}$$

where the components of the vector-function $\theta_r$ are again logarithmic quasipolynomials. We choose in (7.11) any $c_r, \tilde{c}_r$ with $\begin{pmatrix} z_r \\ u_r \end{pmatrix}(1) = 0$. Then, applying Poincaré's theorem, for sufficiently small $\varepsilon$ we obtain a (possibly multiple-valued) analytic in an open neighborhood of $\gamma$ solution of the system $\mathcal{S}_\varepsilon$, given by $z(w :) = \sum_{r=1}^{\infty} z_r(w)\varepsilon^r$, $u(w) := \sum_{r=1}^{\infty} u_r(w)\varepsilon^r$. The uniform convergence in Poincaré's theorem implies that this solution is not single-valued along $\gamma$, because the first term in its expansion $\begin{pmatrix} z_1 \\ u_1 \end{pmatrix}\varepsilon$ is not single-valued along $\gamma$. As the system (7.6) is obtained from (7.1) by scaling of the independent variable $w$, we conclude that there exists a nonsingle-valued solution for (7.1) in some annulus. We get a contradiction with the assumptions of Theorem 3.5, which proves the lemma. q.e.d.

**End of the proof of Theorem 7.1.** Let $k_1 \geq 1$ be the smallest positive eigenvalue of the matrix $L$ (which exists by the assumption), and $k_2 \neq k_1$ be the second eigenvalue (not necessarily positive).

Suppose first $k_1 = 1$. Then we claim that $K_1 = 0$ in (7.8), and one can put $\begin{pmatrix} z_1 \\ u_1 \end{pmatrix} = 0$. Indeed, the system (7.8) implies the scalar inhomogeneous Euler equation

$$z_1'' = \frac{p_{00} + 1}{w} z_1' + \frac{q_{10}}{w^2} z_1 + \frac{q_{01}}{w}, \tag{7.12}$$

for which the basic solutions of the homogeneous equation are some single-valued rational functions of the form const $\cdot\, w$ and const $\cdot\, w^{k_2}$. Then it is straightforward to check that the variation of constants gives a partial solution containing two terms of the form const $\cdot\, w$ and const $\cdot\, w \ln w$, and that the second term is non-zero (and hence not single-valued) iff $K_1 \neq 0$. Proceeding now as in the proof of Lemma 7.4, we see that the possibility $K_1 \neq 0$ contradicts the assumptions of Theorem 3.5, and so $K_1$ must vanish. Hence, for $r = 1$ in (7.4) one can simply put $h_1 := 0$.

If $k_2$ is not positive, we may repeat the proof of the proposition in the nonresonant case, as there are no more obstructions to solve equations (7.4). If $k_2$ is a positive integer, we return to Poincaré's Small Parameter method and analyze it simultaneously with system (7.4). As $K_1 = 0$, we put $\begin{pmatrix} z_1 \\ u_1 \end{pmatrix} = 0$ and $h_1 = 0$ in (7.4). Then, using the expansions for $z^\varepsilon(w)$, $u^\varepsilon(w)$, $P(z, \varepsilon w), Q(z, \varepsilon w)$ and collecting terms with $\varepsilon^r$ for $r = 2$ in (7.4), we have

$$\begin{pmatrix} z_2' \\ u_2' \end{pmatrix} = \frac{1}{w} L \begin{pmatrix} z_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ K_2 \end{pmatrix} \cdot w, \tag{7.13}$$

where $L, K_2$ are as in (7.4) (more precisely we substitute the values $a_1 = 0$ and $b_1 = 0$, found in the previous step, into $K_2$). We consider (7.13), again, as an inhomogeneous Euler equation. The basic solutions are $\text{const} \cdot w$ and $\text{const} \cdot w^{k_2}$. If $r = k_2 = 2$ is the resonant integer, we apply the variation of constants and conclude, in the same way as for the resonant value $r = k_1 = 1$, that $K_2 \neq 0$ contradicts the assumptions of Theorem 3. We may then put $h_2 := 0$ in (7.4) and the rest of the proof repeats that of the proposition in the nonresonant case, as no more resonant integers can exist. If, otherwise, $k_2 > 2$ and hence $r = 2$ is not a resonant integer, one can check that the variation of constants gives a partial solution of the form $\begin{pmatrix} z_2 \\ u_2 \end{pmatrix} = h_2 w^2$, where $h_2$ is a constant vector. It is easy to see that the fact that $h_2 w^2$ is a solution of (7.13) implies that $h_2$ is a (unique!) solution of (7.4). It is then straightforward to check that, proceeding further with the small parameter method and gathering terms with $\varepsilon^3$, one has, in the same spirit as before,

$$\begin{pmatrix} z_3' \\ u_3' \end{pmatrix} = \frac{1}{w} L \begin{pmatrix} z_3 \\ u_3 \end{pmatrix} + \begin{pmatrix} 0 \\ K_3 \end{pmatrix} \cdot w^2, \tag{7.14}$$

where $L, K_3$ are as in (7.4) (more precisely, one has to substitute the values $a_1, b_1, a_2, b_2$, found on the previous steps, into $K_3$). The latter follows from the fact that the second term $h_2 w^2 \varepsilon^2$ in the small parameter expansion agrees with the solution $h_2$ of (7.4) for $r = 2$. In the same way as before, we conclude now that if $k_2 = 3$, then $K_3 = 0$ and we set $h_3 = 0$ in (7.4), in order to avoid a contradiction with the assumptions of Theorem 3.5. We then repeat the proof as in the nonresonant case. Otherwise, we again obtain a partial solution $\begin{pmatrix} z_3 \\ u_3 \end{pmatrix} = h_3 w^3$, where $h_3$ is a constant vector, satisfying (7.4) for $r = 3$.

We continue with the similar arguments until we reach the step $r = k_2$, to get $K_{k_2} = 0$, $h_{k_2} = 0$ in (7.4) and then repeat the proof as in the nonresonant case. This completes the case $k_1 = 1$. The proof in the case $k_1 > 1$ uses the same arguments as above and is completely analogous. q.e.d.

Thus Theorem 3.5 is finally proved. Theorem 3.5 and Proposition 6.10 now imply Theorem 3.6.

## 8. Analytic continuation and infinitesimal automorphisms

It was explained in Section 2 that the monodromy of a mapping associated with a nonminimal pseudospherical hypersurface is given by some $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$. This allows us to obtain in this section a useful representation of the infinitesimal automorphism algebra $\mathfrak{hol}(M,p)$ for $p \in X$ of a nonminimal pseudospherical hypersurface. Combining this representation with Theorem 3.4, we will prove in the next section the Dimension Conjecture.

*Proof of Theorem 3.7.* Fix a collection $\{p, U, \mathcal{F}_0, \mathcal{F}, \mathcal{Q}\}$, where $p \in M$ is a Levi-nondegenerate point, $\mathcal{Q} \subset \mathbb{CP}^n$ a nondegenerate hyperquadric, $\mathcal{F}_0 : (\mathbb{C}^n, p) \longrightarrow (\mathbb{CP}^n, p')$ a biholomorphic mapping with $\mathcal{F}_0(M) \subset \mathcal{Q}$, and $U$ is an open neighbourhood of the origin such that $\mathcal{F}_0$ extends in $U \setminus X$ to a (multiple-valued) locally biholomorphic mapping $\mathcal{F}$ into $\mathbb{CP}^n$ in the sense of Weierstrass. We denote by $M^+, M^-$ the two sides of $M \setminus X$ and assume, without loss of generality, that $p \in M^+$. Fix an element $L \in \mathfrak{hol}\,(M, 0)$ and consider the (connected) flow $\psi_t : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$, $t \in \mathbb{R}$, $\psi_0 = \mathrm{Id}$, generated by $\mathrm{Re}\, L$. Note that any local automorphism $\psi_t$ must preserve the complex hypersurface $X$, and so we may assume that $\psi_t(M^+) \subset M^+$. For $p$ sufficiently close to 0, we may suppose that $\psi_t$ with sufficiently small $t$ are defined in a neighbourhood of $p$ and consider their push-forwards

$$\tau_t := \mathcal{F}_0 \circ \psi_t \circ \mathcal{F}_0^{-1}.$$

Then $\tau_t$ is a flow of local CR-automorphisms of $\mathcal{Q}$ at $p' = \mathcal{F}_0(p)$ and, according to [**12**], $\tau_t \in \mathrm{Aut}\,(\mathcal{Q})$. It is also shown in [**12**] that $\mathrm{Aut}\,(\mathcal{Q})$ is a maximally totally real subgroup of $\mathrm{Aut}(\mathbb{CP}^n)$. Note that the correspondence $\psi_t \to \tau_t$ is injective w.r.t. the flows. Now let us consider the analytic mappings $\mathcal{F}^t := \mathcal{F} \circ \psi_t$ in $U_t \setminus X$ for a sufficiently small polydisc $U_t \subset U$, centred at 0. It is easy to see from the definition of $\mathcal{F}^t$ that its germ at $p$ also maps $(M, p)$ into $\mathcal{Q}$, and if $\sigma$ is the monodromy matrix associated with $\mathcal{F}$ then $\mathcal{F}^t$ has the same monodromy matrix $\sigma$. On the other hand, (2.8) shows that the monodromy of $\mathcal{F}^t$ is given by the matrix $\tau_t \circ \sigma \circ \tau_t^{-1}$ with $\tau_t$ being exactly the push-forward of $\psi_t$. Hence,

$$\sigma = \tau_t \circ \sigma \circ \tau_t^{-1}.$$

Therefore, the push-forward of the automorphisms $\tau_t$ belong to the subgroup $C \subset \mathrm{Aut}(\mathcal{Q})$ that consists of elements of $\mathrm{Aut}(\mathcal{Q}) \subset \mathrm{Aut}(\mathbb{CP}^n)$, commuting with the element $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$. The subgroup $C$ is the intersection of the centralizer $Z(\sigma)$ (see [**52**]) of the element $\sigma \in \mathrm{Aut}\,(\mathbb{CP}^n)$ with the totally real subgroup $\mathrm{Aut}(\mathcal{Q}) \subset \mathrm{Aut}(\mathbb{CP}^n)$. Its tangent algebra is $c = z(\sigma) \cap \mathfrak{hol}\,(\mathcal{Q}, p')$, where $z(\sigma)$ is the tangent algebra to $Z(\sigma)$ (we also call it the *centralizer of $\sigma$).* The above arguments imply the existence of an injective embedding of $\mathfrak{hol}\,(M, p)$ into the algebra $c$.

q.e.d.

As an application we obtain

**Corollary 8.1.** *Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and $\dim \mathfrak{hol}\,(M, 0) \geq 5$. Then either (i) $M$*

*is Levi-flat, or (ii) $(M,0)$ is spherical, or (iii) $M$ is holomorphically equivalent to a hypersurface of class $\mathcal{P}_0$ such that its monodromy operator $\sigma$ is the identity (in other words, the associated mapping $\mathcal{F}$ is single-valued).*

*Proof.* We consider several cases depending on the Levi form of $M$.

If $M$ is Levi-flat, then $\dim \mathfrak{hol}\,(M,0)=\infty$, see [**4**].

If $M$ is Levi nondegenerate at 0, then the classical results in [**43**] and [**12**] imply that $\dim \mathfrak{hol}\,(M,0)\le 8$. Further analysis in [**6**] shows that $\dim \mathfrak{aut}\,(M,0)\le 1$, unless $(M,0)$ is spherical. Combining this with the classification of E. Cartan [**10**] of homogeneous hypersurfaces in $\mathbb{C}^2$, we obtain $\dim \mathfrak{hol}\,(M,0)\le 3$, if $M$ is Levi-nondegenerate and is not spherical at zero.

If $M$ is Levi-degenerate at 0, but not Levi-flat, the hypersurface $M$ can either be of finite type at 0 (see [**4**] for various definitions of type), which is equivalent to its minimality, or $M$ can be of infinite type, which is equivalent to its nonminimality. Some generalizations of Poincaré-Chern-Moser arguments provide the estimate $\dim \mathfrak{hol}\,(M,0)\le 4$ in the finite type case (e.g., [**29**]). Thus we may assume $M$ is nonminimal at 0. Let $\Sigma\subset M$ be the set of points where the Levi form is degenerate. If $\Sigma\ne X$ near the origin, then, since $X$ is the only complex hypersurface contained in $M$ in a sufficiently small neighbourhood of the origin, there exist finite type Levi degenerate points in $M$, arbitrarily close to 0. Applying the bounds from [**29**], we obtain again $\dim \mathfrak{hol}\,(M,0)\le 4$. Thus, we may assume that $M\setminus X$ is Levi-nondegenerate in a sufficiently small neighbourhood of the origin. The inequality $\dim \mathfrak{hol}\,(M,0)\ge 5$ implies that for a Levi-nondegenerate point $p\in M\setminus X$ its infinitesimal automorphism algebra has dimension at least 5. Applying again [**6**] and [**10**], we conclude that $M\setminus X$ is spherical and therefore it is biholomorphically equivalent to some $\tilde{M}\in\mathcal{P}_0$. Thus, it remains to consider only the case when $M\in\mathcal{P}_0$. Theorem 3.7 gives

$$\dim \mathfrak{hol}\,(M,0)\le \dim_{\mathbb{C}} z(\sigma), \tag{8.1}$$

where $\sigma$ is the monodromy operator for $M$ ($\sigma$ can be interpreted as a $3\times 3$ matrix, defined up to scaling). Centralizers of elements of $GL(3,\mathbb{C})$ can be easily analyzed, using the Jordan normal form, and it is not difficult to see that for all nonscalar matrices the centralizer has dimension at most 5. Taking the scaling into account, we have $\dim_{\mathbb{C}} z(\sigma)\le 4$, unless $\sigma=\mathrm{Id}$. q.e.d.

It follows immediately follows from Corollary 8.1 that Theorem 3.8 implies the Strong Dimension Conjecture. The following proposition gives the answer in the case when $\mathcal{F}$ is single-valued and extends to $X$.

**Proposition 8.2.** *Let $M\subset\mathbb{C}^2$ be of class $\mathcal{P}_0$, and $U$ be the associated neighbourhood. Assume, in addition, that the associated mapping $\mathcal{F}$*

*extends to the complex locus $X$ holomorphically. Then* $\mathfrak{hol}\,(M,0)$ *can be injectively embedded into the stability algebra* $\mathfrak{aut}\,(S^3,o')$ *for some point* $o'\in S^3$*. In particular, dim* $\mathfrak{hol}\,(M,0)\leq 5$.

*Proof.* First note that $\mathcal{F}(X)$ is a locally countable union of locally complex analytic sets [**13**]. On the other hand, $\mathcal{F}(X)$ is connected and $\mathcal{F}(X)\subset S^3$, so that we conclude that $\mathcal{F}(X)=\{o'\}$ for some point $o'\in S^3$. Choose now a point $q\in M^+$ ($M^+,M^-$ are the sides of $M\setminus X$) and a local flow $\psi_t$ of local automorphisms of $M$ near the origin, $\psi_t(M^+)\subset M^+$. Shrinking $U$ if necessary, we may suppose that $\psi_t$ is defined in $U$. Arguing as in the proof of Theorem 3.7, we may consider the push-forward $\tau_t:=\mathcal{F}\circ\psi_t\circ\mathcal{F}^{-1}$ defined in a neighbourhood of the point $q'=\mathcal{F}(q)$ (we choose the element of $\mathcal{F}^{-1}$ with $\mathcal{F}^{-1}(q')=q$). Since $\psi_t(M)\subset M$, we have $\tau_t(S^3)\subset S^3$, so $\tau_t$ extends to an element of $\mathrm{Aut}\,(S^3)\subset\mathrm{Aut}\,(\mathbb{CP}^2)$ (see [**12**]). Then for points $z\in\mathbb{C}^2$, close to $q$, we have $\mathcal{F}\circ\psi_t(z)=\tau_t\circ\mathcal{F}(z)$. By uniqueness the latter equality holds for all $z\in U$. Therefore, $\mathcal{F}(\psi_t(0))=\tau_t(\mathcal{F}(0))$ and, since $0\in X$, $\psi_t(X)\subset X$, and $\mathcal{F}(X)=\{o'\}$, we conclude that $\tau_t(o')=o'$, and so $\tau_t$ stabilize the point $o'$. Applying this to a local flow $\psi_t$, generated by $\mathrm{Re}\,L$ for some $L\in\mathfrak{hol}\,(M,0)$, we conclude that the flow $\tau_t:=\mathcal{F}\circ\psi_t\circ\mathcal{F}^{-1}$ extends to a flow $\tau_t\in\mathrm{Aut}\,(S^3)$ with $\tau_t(o')=o'$, and then for the corresponding vector field $L'\in\mathfrak{hol}\,(S^3,q')$ we have $L'(o')=0$. As the correspondence $\psi_t\longrightarrow\tau_t$ is injective w.r.t. a flow $\psi_t$, the proposition follows. q.e.d.

**Corollary 8.3.** *Theorem 3.8 holds true for any hypersurface* $M\in\mathcal{P}_0$*, except, possibly, the case of a hypersurface with a single-valued associated mapping* $\mathcal{F}$*, which does not extend holomorphically to the complex locus* $X$*. In particular, the Strong Dimension Conjecture holds true for any 1-nonminimal at the origin smooth real-analytic hypersurface* $M\subset\mathbb{C}^2$.

## 9. Solution of the Dimension Conjecture

In this section we complete the proof of the Dimension Conjecture. In view of Section 8, it remains to treat the case of an $m$-nonminimal hypersurface $M\in\mathcal{P}_0$ with a single-valued mapping $\mathcal{F}:\,U\setminus X\longrightarrow\mathbb{CP}^2$ associated with $M$, which does not extend to $\{w=0\}$ holomorphically.

Consider the Lie algebra $\mathfrak{g}=\mathfrak{hol}(M,0)$ and its complexification $\mathfrak{h}=\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\otimes\mathbb{C}$. Fix a Levi nondegenerate point $p\in M$, for which all vector fields $L\in\mathfrak{g}$ are defined, and for a vector field $L\in\mathfrak{g}$ consider, as in the proof of Theorem 3.7, its push-forward $L^*\in\mathfrak{hol}(\mathbb{CP}^2)$. Then we obtain a well-defined push-forward $(\mathfrak{g}^*,\mathfrak{h}^*)$ for the pair $(\mathfrak{g},\mathfrak{h})$. Here $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are a real and a complex Lie subalgebras of $\mathfrak{hol}(\mathbb{CP}^n)$ respectively, naturally isomorphic to the algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. It follows from our construction that the pulled-back algebra $\mathcal{F}^{-1}\circ\mathfrak{h}^*$ coincides with $\mathfrak{h}$, in particular, all vector fields from the well-defined in $U\setminus X$ algebra

$\mathcal{F}^{-1}\circ\mathfrak{h}^*$ extend to $X$ holomorphically. We also note that a projective change of coordinates in $\mathbb{CP}^2$, given by $\tau\in \mathrm{PGL}(3,\mathbb{C})$, replaces the mapping $\mathcal{F}$ with the mapping $\tau\circ\mathcal{F}$. At the same time, $\tau$ conjugates the Lie algebra $\mathfrak{hol}(\mathbb{CP}^n)\simeq\mathfrak{sl}(3,\mathbb{C})$, and $\mathfrak{h}^*$ changes accordingly (see Section 2).

We now need the following statement.

**Proposition 9.1.** *Fix an affine chart $V\subset\mathbb{CP}^2$ with the affine coordinates $(z^*,w^*)$. Then the algebra $\mathfrak{h}^*$ cannot contain the 2-dimensional subalgebra, given in $V$ by*

$$span_{\mathbb{C}}\left\{\frac{\partial}{\partial z^*},\frac{\partial}{\partial w^*}\right\}. \tag{9.1}$$

*Proof.* Assume on the contrary that $\mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^*},\frac{\partial}{\partial w^*}\right\}\subset\mathfrak{h}^*$. Take the regular set $U^0\subset U\setminus X$ (see Section 5) and consider $\mathcal{F}$, restricted to $U^0$, as a mapping into $V$. Consider first the case when $\alpha_0(w)\not\equiv 0$ in (3.6). We represent $\mathcal{F}$ as in (5.15) with single-valued $\alpha(w),\beta(w),a(w),b(w),\delta(w)$. Then, applying (5.15), we have

$$\mathcal{F}^{-1}\circ\frac{\partial}{\partial z^*}=T_1(z,w)\frac{\partial}{\partial z}+\frac{a}{\alpha' a-a'\alpha}(z+\delta)\frac{\partial}{\partial w}, \tag{9.2}$$

$$\mathcal{F}^{-1}\circ\frac{\partial}{\partial w^*}=T_2(z,w)\frac{\partial}{\partial z}-\frac{\alpha}{\alpha' a-a'\alpha}(z+\delta)\frac{\partial}{\partial w}. \tag{9.3}$$

Here $T_1(z,w),T_2(z,w)$ are some specific functions, but their exact form is of no importance to us. Since the vector fields in (9.2) and (9.3) extend holomorphically to $X$, the functions $P(z,w)=\frac{a}{\alpha' a-a'\alpha}(z+\delta)$ and $Q(z,w)=\frac{\alpha}{\alpha' a-a'\alpha}(z+\delta)$ are holomorphic near the origin. From this it follows that $\delta(w)\in\mathcal{M}(0)$. Further, letting $a(w)=k(w)\alpha(w)$, we conclude that $k(w)=\frac{P}{Q}\in\mathcal{M}(0)$. Since $Q(z,w)=-\frac{1}{k'\alpha}(z+\delta)$, it follows that $k'\alpha\in\mathcal{M}(0)$, so that $\alpha(w),a(w)\in\mathcal{M}(0)$. Note that $k(w)$ is not a constant, as this would contradict (5.17). Thus, by Theorem 3.4, $\mathcal{F}$ extends to $X$ holomorphically, which is a contradiction.

Now consider the case when $\alpha_0(w)\equiv 0$ in (3.6). It follows that $\mathcal{F}=(f,g)$ satisfies

$$f=\alpha z+\beta,\; g=az+b \tag{9.4}$$

for some single-valued meromorphic in $\Delta^*_\epsilon$ functions $\alpha(w),\beta(w),a(w),b(w)$. Then either $\alpha\not\equiv 0$ or $a\not\equiv 0$ (as $\mathcal{F}$ is locally injective). Say, $\alpha\not\equiv 0$, so we set $k(w):=\frac{a(w)}{\alpha(w)}$. Then the fact that $I_1(z,w)=0$ in (5.6) (see Proposition 5.2) yields the special relation $\alpha' a-a'\alpha=0$, which implies that $k$ is a constant. We now apply (9.4) to conclude that the Jacobian of the mapping $\mathcal{F}$ is equal to $\alpha(b'-k\beta')$, and that

$$(9.5)\quad \mathcal{F}^{-1}\circ\frac{\partial}{\partial z^*}=\left(k\frac{\alpha'}{\alpha}\frac{1}{b'-k\beta'}z+\frac{b'}{\alpha}\frac{1}{b'-k\beta'}\right)\frac{\partial}{\partial z}-k\frac{1}{b'-k\beta'}\frac{\partial}{\partial w},$$

$$(9.6)\quad \mathcal{F}^{-1}\circ\frac{\partial}{\partial w^*}=-\left(\frac{\alpha'}{\alpha}\frac{1}{b'-k\beta'}z+\frac{\beta'}{\alpha}\frac{1}{b'-k\beta'}\right)\frac{\partial}{\partial z}+\frac{1}{b'-k\beta'}\frac{\partial}{\partial w}.$$

As both (9.5) and (9.6) extend to $X$ holomorphically, we conclude first that $b'-k\beta'\in\mathcal{M}(0)$ and second, considering the linear combination $F^{-1}\circ\frac{\partial}{\partial z^*}+k\mathcal{F}^{-1}\circ\frac{\partial}{\partial w^*}=\frac{1}{\alpha}\frac{\partial}{\partial z}$, that $\alpha\in\mathcal{M}(0)$. These two conclusions imply $\beta',b'\in\mathcal{M}(0)$ and finally $\beta,b,a\in\mathcal{M}(0)$. Then, by Theorem 3.4, $\mathcal{F}$ extends to $X$ holomorphically, which is again a contradiction. This proves the proposition. q.e.d.

Our next goal is the classification of higher-dimensional Lie subalgebras of $\mathfrak{sl}(3,\mathbb{C})$. We could not find an appropriate reference in the literature, so for the sake of completeness we provide the proof that was suggested to us by Andrey Minchenko. By a *matrix element* $e_{ij}$ we mean a square matrix all of whose entries are zero, except the entry in the $i$-th row and the $j$-th column which equals 1.

**Proposition 9.2.** *Let* $\mathfrak{l}\subset\mathfrak{sl}(3,\mathbb{C})$ *be a complex Lie subalgebra,* $\dim\mathfrak{l}\geq 5$. *Denote by* $\mathfrak{b}_\pm$ *the subalgebras of upper-triangular and lower-triangular elements of* $\mathfrak{sl}(3,\mathbb{C})$ *respectively, and by* $\mathfrak{r}_\pm$ *the subalgebras of zero last row and zero last column elements of* $\mathfrak{sl}(3,\mathbb{C})$ *respectively. Let* $\mathfrak{p}_+=\mathfrak{b}_+\oplus\mathbb{C}e_{21}$, *and* $\mathfrak{p}_-=\mathfrak{b}_-\oplus\mathbb{C}e_{23}$. *Then* $\mathfrak{l}$ *is conjugated in* $\mathfrak{sl}(3,\mathbb{C})$ *to one of the subalgebras* $\mathfrak{b}_+$, $\mathfrak{r}_\pm$, $\mathfrak{p}_\pm$, *or* $\mathfrak{sl}(3,\mathbb{C})$.

*Proof.* In what follows we refer to [**52**] for various facts from the Lie theory. First, consider the case when $\mathfrak{l}$ is solvable. Then, as $\dim\mathfrak{l}\geq 5$, we conclude that $\mathfrak{l}$ is the Borel subalgebra. As the Borel subalgebra is unique, up to a conjugation, we conclude that $\mathfrak{l}$ is conjugated to $\mathfrak{b}_+$. If, otherwise, $\mathfrak{l}$ is not solvable, then its Levi-Malcev decomposition contains a nontrivial semi-simple factor. From the structure theory of semi-simple Lie algebras, any such factor contains a subalgebra, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. It is known that there exist, up to a conjugation, exactly two subalgebras in $\mathfrak{sl}(3,\mathbb{C})$, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$: the first one is $\mathfrak{so}(3,\mathbb{C})\subset\mathfrak{sl}(3,\mathbb{C})$, and the second one is $\mathfrak{sl}(2,\mathbb{C})\subset\mathfrak{sl}(3,\mathbb{C})$, embedded as the left upper $2\times 2$ block, so that we may suppose that, after an appropriate conjugation, $\mathfrak{l}$ contains one of the above subalgebras. Consider first the case of $\mathfrak{so}(3,\mathbb{C})\subset\mathfrak{l}\subset\mathfrak{sl}(3,\mathbb{C})$. Then the subalgebra $\mathfrak{so}(3,\mathbb{C})$ acts on $\mathfrak{sl}(3,\mathbb{C})$ by the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$, restricted onto $\mathfrak{so}(3,\mathbb{C})$. Decomposing $\mathfrak{sl}(3,\mathbb{C})$ into a direct sum of irreducible invariant subspaces for the above action, we get the decomposition

$$\mathfrak{sl}(3,\mathbb{C})=\mathfrak{so}(3,\mathbb{C})\oplus V,$$

where $V$ is the subspace of all symmetric matrices from $\mathfrak{sl}(3,\mathbb{C})$. The subalgebra $\mathfrak{l}$ must be the sum of $\mathfrak{so}(3,\mathbb{C})$ and some of the invariant subspaces, so $\mathfrak{l} = \mathfrak{sl}(3, \mathbb{C})$ or $\mathfrak{l} = \mathfrak{so}(3, \mathbb{C})$. As $\dim \mathfrak{l} \geq 5$, we summarize the $\mathfrak{so}(3, \mathbb{C})$-case with the conclusion $\mathfrak{l} = \mathfrak{sl}(3, \mathbb{C})$.

Consider now the case $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{l} \subset \mathfrak{sl}(3, \mathbb{C})$. Arguing as in the $\mathfrak{so}(3, \mathbb{C})$-case, we obtain the decomposition

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{13} \oplus \mathbb{C}e_{23}) \oplus (\mathbb{C}e_{31} \oplus \mathbb{C}e_{32}) \oplus \mathbb{C}h$$

of $\mathfrak{sl}(3, \mathbb{C})$ into the direct sum of irreducible invariant subspaces of $\mathfrak{sl}(3, \mathbb{C})$ under the adjoint action of $\mathfrak{sl}(3, \mathbb{C})$, restricted onto $\mathfrak{sl}(2, \mathbb{C})$. Here $h = \mathrm{diag}\{1, 1, -2\}$. The algebra $\mathfrak{l}$ is the direct sum of $\mathfrak{sl}(2, \mathbb{C})$ and some of the invariant subspaces. Then, in view of the assumption $\dim \mathfrak{l} \geq 5$, we obtain the following list of distinct decompositions of $\mathfrak{l}$:

$$\begin{aligned}
\mathfrak{l} &= \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{13} \oplus \mathbb{C}e_{23}); \\
\mathfrak{l} &= \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{31} \oplus \mathbb{C}e_{32}); \\
\mathfrak{l} &= \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{13} \oplus \mathbb{C}e_{23}) \oplus \mathbb{C}h; \\
\mathfrak{l} &= \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{31} \oplus \mathbb{C}e_{32}) \oplus \mathbb{C}h; \\
\mathfrak{l} &= \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}e_{13} \oplus \mathbb{C}e_{23}) \oplus (\mathbb{C}e_{31} \oplus \mathbb{C}e_{32}) \oplus \mathbb{C}h.
\end{aligned}$$

This implies the claim of the proposition. q.e.d.

The classification implies

**Proposition 9.3.** *Let $\mathfrak{l}$ be a subalgebra in $\mathfrak{hol}(\mathbb{CP}^2)$ with $\dim \mathfrak{l} \geq 5$. Then there exists an affine chart $V$ with coordinates $(z^*, w^*)$ such that $\mathfrak{l}$ contains the 2-dimensional subalgebra $\mathfrak{a}$, given in $V$ by* (9.1).

*Proof.* Interpreting the commutative Lie algebra $\mathfrak{a}$ of holomorphic vector fields as a subalgebra in $\mathfrak{sl}(3, \mathbb{C})$, we obtain the representation of $\mathfrak{a}$ as $\mathrm{span}_{\mathbb{C}}\{e_{13}, e_{23}\}$ (we use the notation of Proposition 9.2 in what follows). In order to use the classification, given by Proposition 9.2, we assign to each conjugacy in the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ a projective coordinate change in $\mathbb{CP}^2$, and get the corresponding affine chart $V \subset \mathbb{CP}^2$ with the coordinates $(z^*, w^*)$ (see Section 2.5). Note that the subalgebras $\mathfrak{b}_+, \mathfrak{r}_+, \mathfrak{p}_+ \subset \mathfrak{sl}(3, \mathbb{C})$ already contain $\mathfrak{a}$. Further, it is straightforward to check that the matrix $A = e_{31} + e_{12} + e_{23} \in \mathrm{SL}(3, \mathbb{C})$ conjugates the matrices $e_{21}, e_{31} \in \mathfrak{r}_- \cap \mathfrak{p}_-$ with the matrices $e_{13}, e_{23}$ respectively. The latter implies that *any* subalgebra $\mathfrak{l} \subset \mathfrak{sl}(3, \mathbb{C})$ with $\dim \mathfrak{l} \geq 5$ contains, after an appropriate conjugation, the algebra $\mathfrak{a}$. This proves the proposition. q.e.d.

Combined, Propositions 9.1, Proposition 9.3 and Corollary 8.1 yield

**Corollary 9.4.** *Let $M \in \mathcal{P}_0$, and the associated mapping $\mathcal{F}$ does not extend to $X$ holomorphically. Then $\dim \mathfrak{hol}(M, 0) \leq 4$.*

Corollary 9.4 immediately implies the proof of Theorem 3.8. Combining with Corollary 8.1, we obtain also Theorem 3.9. Theorem 3.10 follows from a combination of Corollary 9.4, Theorem 3.9 and Corollary

8.1. Finally, Theorem 3.11 follows from the fact the any Lie algebra of dimension $\leq 3$ is contained in $\mathfrak{su}(2,1)$ (see, e.g., [**52**]), and in the case $4 \leq \dim \mathfrak{hol}(M,0) < \infty$ $M$ needs to be spherical at its generic point and the embedding into $\mathfrak{hol}(S^3,o)$ is immediate. The bound $\dim \mathfrak{hol}(M,0) \leq 5$ in the nonspherical case follows from Theorem 3.10.

All the results of the paper are completely proved now.

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